MATHEMATICS

STUDY MATERIAL

CIRCLE

IIT-JEE

श्रमेव जयते
THE NARAYANA GROUP

NARAYANA INSTITUTE OF CORRESPONDENCE COURSES
FNS HOUSE, 63 KALU SARAI MARKET
SARVAPRIYA VIHAR, NEW DELHI-110016
PH.: (011) 32001131/32/50 • FAX : (011) 41828320
Website : www.narayanaicc.com
E-mail : info@narayanaicc.com

PREFACE

Dear Student,

Heartiest congratulations on making up your mind and deciding to be an engineer to serve the society.

As you are planning to take various Engineering Entrance Examinations, we are sure that this **STUDY PACKAGE** is going to be of immense help to you.

At NARAYANA we have taken special care to design this package according to the **Latest Pattern of IIT-JEE,** which will not only help but also guide you to compete for IIT-JEE, AIEEE & other State Level Engineering Entrance Examinations.

The salient features of this package include :

- Power packed division of units and chapters in a scientific way, with a correlation being there.
- Sufficient number of solved examples in Physics, Chemistry & Mathematics in all the chapters to motivate the students attempt all the questions.
- All the chapters are followed by various types of exercises, including Objective - Single Choice Questions, Objective - Multiple Choice Questions, Comprehension Type Questions, Match the Following, Assertion-Reasoning & Subjective Questions.

These exercises are followed by answers in the last section of the chapter including Hints & Solutions wherever required. *This package will help you to know* what to study, how to study, time management, your weaknesses and improve your performance.

We, at NARAYANA, strongly believe that quality of our package is such that the students who are not fortunate enough to attend to our Regular Classroom Programs, can still get the best of our quality through these packages.

We feel that there is always a scope for improvement. We would welcome your suggestions & feedback.

Wish you success in your future endeavours.

THE NARAYANA TEAM

ACKNOWLEDGEMENT

While preparing the study package, it has become a wonderful feeling for the NARAYANA TEAM to get the wholehearted support of our Staff Members including our Designers. They have made our job really easy through their untiring efforts and constant help at every stage.

We are thankful to all of them.

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CONTENTS

CIRCLE

CIRCLE

IIT- JEE SYLLABUS

Equation of a circle in various forms, Equations of tangent, Normal and chord, Parametric equations of a circle, Intersection of a circle with a straight line or a circle, Equation of a circle through the points of intersection of two circles and those of a circle and a straight line, Locus Problems.

CONTENTS

INTRODUCTION

Many objects in daily life, which are round in shape like rings, bangles, wheels of a vehicle etc. are the examples of a circle. In terms of mathematics, circle is an important locus of a point in two dimensional coordinate geometry. Some times circle is also called as a part of conic section. In this chapter we deal circle as an independent topic.

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1. DEFINITION OF CIRCLE

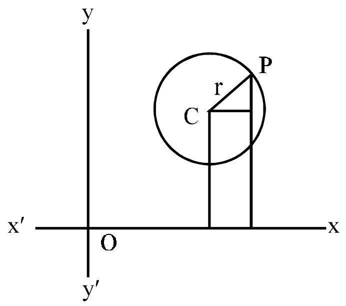
A circle is the locus of a point which moves in such a way that its distance from a fixed point is constant. The fixed point is called the centre of the circle and the constant distance, the radius of the circle.

2. EQUATION OF THE CIRCLE IN VARIOUS FORMS

(A) Centre - Radius Form

To find the equation of circle when the centre and radius are given.

Let r be the radius and C (α, β) the centre and P any point on the circle whose coordinates are (x, y).



Then $CP^2 = r^2$

i.e. $(x - \alpha)^2 + (y - \beta)^2 = r^2$

this equation reduces to $x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - r^2 = 0$ ...(i)

General equation of 2nd degree :

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad ...(ii)$$

Let this equation represents a circle with centre (α, β) and radius r.
Hence equation (i) & (ii) represent same circle
Comparing (i) & (ii) we get,

$$\left.\begin{array}{ll} a = b & \Rightarrow \text{coefficient of } x^2 = \text{coefficient of } y^2 \\ h = 0 & \Rightarrow \text{coefficient of } xy = 0 \end{array}\right\}$$

These are required condition, for general equation of 2nd degree represents a circle
For general equation of a circle put a = b = 1, h = 0 in equation (ii)
Hence general equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(iii)
Centre $(\alpha, \beta) = (-g, -f)$

$\therefore \quad c = g^2 + f^2 - r^2 \qquad \therefore \quad r^2 = g^2 + f^2 - c$

$\therefore \quad$ Radius = $\sqrt{g^2 + f^2 - c}$

Point circle:

The equation $(x - a)^2 + (y - b)^2 = 0$ represents a point circle since its radius is zero. So the circle converts into just a point.

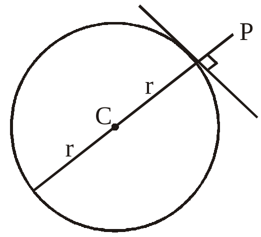
REMARK

(i) $g^2 + f^2 - c > 0 \Rightarrow$ equation represents a real circle

(ii) $g^2 + f^2 - c = 0 \Rightarrow$ equation represents a point circle

(iii) $g^2 + f^2 - c < 0 \Rightarrow$ equation represents an imaginary circle

Tips for Solving Questions



(1) The general equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ which obviously involves three constants g, f and c. ***So to completely specify a circle, 3 conditions are required***. These three conditions may be in any form i.e. 3 points on the circle may be given or centre and radius may be given or 2 points and radius may be given etc. Through any three non-collinear given points, a unique circle can be drawn Through two or less than two given points, infinite number of circles can be drawn and if a circle passes through more than 3 points, then the points must satisfy some condition(s) i.e. a circle cannot always be drawn passing through any 4 points or more in space.

(2) The least and the greatest distances of a point P from a circle with centre C & radius r are $|PC - r|$ and PC + r respectively.

Illustration 1 : ***Find the centre and radius of the circle $3x^2 + 3y^2 - 12x + 6y + 11 = 0$***

Solution : Dividing by 3, we get

$$x^2 + y^2 - 4x + 2y + \frac{11}{3} = 0$$

i.e., $$(x-2)^2 + (y+1)^2 = \frac{4}{3} = \left(\frac{2}{\sqrt{3}}\right)^2.$$

The centre is (2, – 1) and radius $\frac{2}{\sqrt{3}}$.

EXERCISE - 1

1. Find the equations of the circles which have the origin for centre and which pass through the given points-

   (i) (4, – 3) (ii) (–8, – 15) (iii) $(\sqrt{2}, -7\sqrt{2})$;

   (iv) $(4\sqrt{5}, -8)$ (v) (a – b, a + b)

2. Find the equations of the circles whose centres and radii are given -

   (i) (0, 2), 2 ; (ii) (8, 15), 12 (iii) (– 3, – 2) 4

   (iv) (3, 4), 6 (v) (a + b, a – b), $(\sqrt{2a^2 + 2b^2})$.

3. Find the centre and radius of each of the circles -

   (i) $x^2 + y^2 + 4x - 6y - 3 = 0$ (ii) $x^2 + y^2 - 10x + 12y - 20 = 0$;

   (ii) $x^2 + y^2 + 8x + 2y + 8 = 0$ (iv) $3x^2 + 3y^2 + 6x - 9y + 1 = 0$

   (v) $x^2 + y^2 - 4ax + 10by + 4a^2 + 16b^2 = 0$

4. The coordinates of C, the centre of a circle, and of A, a point on the circle, are given; find the equation of the circle in each case -

   (i) C (3, 5), A(1, 2) (ii) C(– 1, 4) A (5, 7)
   (iii) C (4, – 9), A(0, 0) (iv) C (–2, – 6), A (–1, – 6)
   (v) C(a, – b), A (b, a)

5. Find the greatest and the least distance from the origin to point on the circle $x^2 + y^2 - 6x - 8y = 11$.

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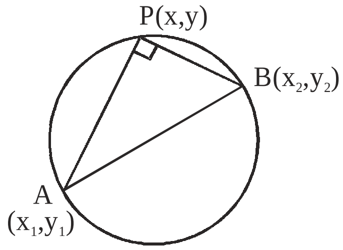
(B) Diametrical Form :

If (x_1, y_1) and (x_2, y_2) are the ends of a diameter of a circle, then the equation of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

This follows from the fact that the angle in a semicircle is a right angle. Thus if P(x, y) be an arbitrary point on the circle other than $A(x_1, y_1)$ and $B(x_2, y_2)$, then $\angle APB$ is a right angle and hence the product of the slopes of AP and BP will be – 1, i.e.



$$\left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

or $$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Tips for Solving Questions

1. The above said circle i.e. $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ is the circle of smallest radius passing through (x_1, y_1) and (x_2, y_2)

2. Centre of the circle given by $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ and its radius will be $\frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Illustration 2 : ***Find the equation of the circle having the origin and (2, 3) as the ends of a diameter.***

Solution : Here the two ends of a diameter are (0, 0) and (2, 3). Hence the required equation of the circle is

$$(x - 0)(x - 2) + (y - 0)(y - 3) = 0 \quad \text{or} \quad x^2 + y^2 - 2x - 3y = 0$$

EXERCISE - 2

1. Find the equation of the circle on AB as diameter, the coordinates of A and B are (4, 8), (10, 2)

2. ABCD is a square whose side is a, taking AB and AD as axes, prove that equation to the circle circumscribing the square is $x^2 + y^2 = a(x + y)$.

(C) Three point (Non-Collinear) Form :

Method I

If three non-collinear points are given, then through them passes a unique circle. In fact, if the point are (x_1, y_1), (x_2, y_2) and (x_3, y_3), then the equation of the circle will be

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

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Method II

A unique circle can be drawn through three non-collinear points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. The circle is the circumcircle of the triangle ABC.

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{........(1)}$$

Since the circle passes through A, B & C, the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) satisfy (I); hence

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \qquad \text{.........(2)}$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \qquad \text{.........(3)}$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \qquad \text{.........(4)}$$

There are thus three linear equations in which the unknowns are g, f and c. The solution of these equations gives the appropriate values of g, f and c which, when substituted in (1), yield the equation of the circle from which the coordinates of the centre and the value of the radius can be derived.

Method III

Here we describe a general and simple method for deriving the equation of a circle through three given points which is less obvious than that which depends on solving three equations for g, f and c. It is based on the geometrical theorem that the centre of a circle is the intersection of the perpendicular bisectors of two non-parallel chords.

Find the equations of two chords from the three given points of the circle. Find the equations of perpendicular bisectors of these chords. Find the point of intersection of these perpendicular bisectors. This becomes the centre of the circle. Since three points are given on the circle, radius can easily be found.

Method IV

The given points (non-collinear) are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Let

$$lx + my + n = 0$$

denote the equation of the line joining A and B ; then

$$lx_1 + my_1 + n = 0,\ lx_2 + my_2 + n = 0 \qquad \text{........(1)}$$

Consider the equation

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda\,(lx + my + n) = 0 \qquad \text{........(2)}$$

where λ is an unspecified constant at present. The equation (2), when expanded, has the form of the general equation of a circle i.e. the coefficients of x^2 and y^2 are same and there is no term in xy. Further, from (1), it is clear that the coordinates of A and of B satisfy (2); accordingly, (2) is the equation of any circle passing through A and B. If the circle passes through C, the coordinates (x_3, y_3) must satisfy (2) ; hence

$$(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2) + \lambda(lx_3 + my_3 + n) = 0$$

from which the appropriate value of λ is obtained. Put this value of λ in equation (2) to get the equation of circle.

This method comes under Family of Circles which will be discussed later.

Method V

Obviously the three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ make a triangle (since the points are non collinear). The circumcentre of the triangle ABC is the point which is equi-distant from A, B and C. Let this point be (h, k).

Obviously $(x_1 - h)^2 + (y_1 - k)^2 = (x_2 - h)^2 + (y_2 - k)^2 = (x_3 - h)^2 + (y_3 - k)^2$

Solving these equations, we get (h, k) i.e. the centre of the circle. Since the three points on the circle are already known we can easily find the radius.

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Illustration 3 : ***Find the equation of the circle passing through the points (0, 1), (2, 3) and (– 2, 5).***

Solution : Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the required circle.

By substituting the coordinates of the given points in the equation of the circle, we get

$$1 + 2f + c = 0 \qquad ....(1)$$

$$13 + 4g + 6f + c = 0 \qquad ....(2)$$

$$29 - 4g + 10f + c = 0 \qquad ....(3)$$

Solving (1), (2) and (3), we get

$$f = -\frac{10}{3},\ g = \frac{1}{3} \quad \text{and} \quad c = \frac{17}{3}$$

The required equation is $3x^2 + 3y^2 + 2x - 20y + 17 = 0$

Illustration 4 : ***Find the equation of the circle through the points A (12, 4), B(8, 12) and C(–6, – 2).***

Solution : **Method I**

Substitute the coordinates of A, B and C in succession in the general equation; then

$$24g + 8f + c = -160, \qquad ....(1)$$

$$16g + 24f + c = -208, \qquad ....(2)$$

$$-12g - 4f + c = -40 \qquad ....(3)$$

Subtract (2) from (1) and divide by 8, then

$$g - 2f = 6 \qquad ....(4)$$

Subtract (3) from (1) and divide by 12, then

$$3g + f = -10 \qquad ....(5)$$

From (4) and (5), $g = -2$ and $f = -4$. Substitute these in (iii); then $c = -80$. The circle is : $x^2 + y^2 - 4x - 8y - 80 = 0$;

its centre is at (2, 4) and its radius is 10.

Method II

The mid-point of AB is (10, 8) ; the gradient of AB is $(12 - 4)/(8 - 12)$ or -2; the perpendicular bisector is, then

$$y - 8 = \frac{1}{2}(x - 10) \text{ or } x - 2y + 6 = 0 \qquad ....(6)$$

The mid-point of BC is (1, 5); the gradient of BC is 1 ; the perpendicular bisector is

$$x + y - 6 = 0 \qquad ....(7)$$

The solution of (6) and (7) is ; $x = 2$, $y = 4$; hence the centre, S, is (2, 4). Also, $r^2 = SA^2 = 10^2 = 100$.

The circle is then $(x - 2)^2 + (y - 4)^2 = 100$, $x^2 + y^2 - 4x - 8y - 80 = 0$.

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Illustration 5 : ***Find the equations of the circles, each of radius 5, which cut the x-axis at A (1, 0) and B (7, 0).***

Solution : **Method I**

Substitute the coordinates of A and B, in succession, in the general equation

$$x^2 + y^2 + 2gx + 2fy + c = 0;$$

then, $1 + 2g + c = 0,\ 49 + 14g + c = 0.$

The solution of these equations gives:

$$g = -4,\ c = 7$$

The radius of the circle is given by

$$r^2 = g^2 + f^2 - c \text{ or } 25 = 16 + f^2 - 7 \text{ whence } f^2 = 16 \text{ and } f = \pm 4.$$

The circle are $(x - 4)^2 + (y \pm 4)^2 = 25$

or $x^2 + y^2 - 8x \pm 8y + 7 = 0.$

C(4, 4) and C_1 (4, – 4) are the centres of the circles.

Method II

The centre lies on the perpendicular bisector of AB; the mid-point M, of AB is (4, 0) and, since A and B lie on the x-axis, the perpendicular bisector is the line x = 4. Accordingly, the coordinates of the centre can be written as (4, k) where k is to be determined.

Now, $AM^2 + k^2 = r^2 = 25$; also AM = 3; hence $k^2 = 16$ and $k = \pm 4$. The equations of the circles are then

$$(x - 4)^2 + (y \pm 4)^2 = 25$$

which leads to (i) above and the coordinates of C and C_1.

Note :

If three points are collinear, no real circle can be drawn through them. The system of three equations will then be inconsistent.

EXERCISE - 3

1. Find the centre and radius of the circle passing through the three given points -

   (i) (4, – 5), (–1, 2), (11, 0) (ii) (–2, 5), (4, 9), (2, – 1)

   (iii) (9, 5), (1, 9), (11, – 1) (iv) (4, 6), (8, 12), (–2, 10)

   (v) (0, – 5), (2, – 1), (–4, 7) (vi) (5, 3), (–7, – 1), (5, – 1)

2. Find the equations of the two circles, of radius 5, passing through (2, 0) and (10, 0).

3. A circle cuts the x-axis at A(4, 0) and B(8, 0) and the y-axis at C(0, – 2) and D(0, k); find its equation and the value of k.

4. Find the equation of the circle through the origin and (0, 4) whose centre lies on the line $2x - y + 4 = 0$

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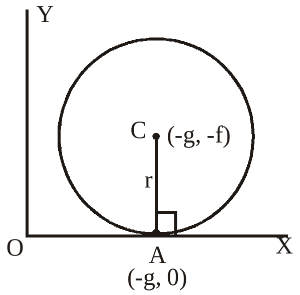
(D) GENERAL EQUATION OF A CIRCLE TOUCHING THE AXES OF CO-ORDINATES:

1. Touching X - Axis:

In fig. a circle, with centre C(– g, – f) is shown touching the x-axis at A, the general equation of the circle being



$$x^2 + y^2 + 2gx + 2fy + c = 0;$$

the radius is $\sqrt{(g^2 + f^2 - c)}$

OX is the tangent at A and AC is the normal at A; the radius of the circle is equal to AC and hence

$$AC = r = |-f|$$

squaring, $r^2 = f^2$

$\Rightarrow \quad g^2 + f^2 - c = f^2 \quad \Rightarrow \quad c = g^2$ ....(1)

The general equation of the circle touching the x-axis is then

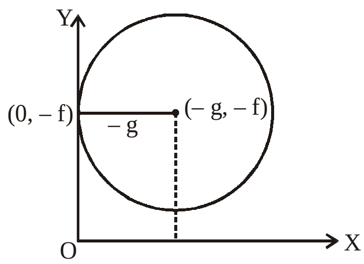
$$x^2 + y^2 + 2gx + 2fy + g^2 = 0$$

or $(x + g)^2 + y^2 + 2fy = 0$ ....(2)

Tips for Solving Questions

If a circle touches x-axis, then by putting y = 0, the remaining equation gives coincident values of x. i.e. the discriminant of the remaining quadratic equation is zero.

2. Touching Y - Axis:



From the figure it is clear that,

$r = |-g| \quad \Rightarrow \quad r^2 = g^2$

$\Rightarrow \quad g^2 + f^2 - c = g^2 \quad \Rightarrow \quad f^2 = c$

So the equation of circle is

$$(x + g)^2 + (y + f)^2 = g^2$$

$\Rightarrow \quad x^2 + y^2 + 2gx + 2fy + f^2 = 0$ ....(3)

$\Rightarrow \quad x^2 + 2gx + (y + f)^2 = 0$

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Tip for Solving Questions

If a circle touches y-axis, then by putting x = 0, the remaining equation gives coincident values of y. i.e. the discriminant of the remaining quadratic equation is zero.

3. Touching both axes:

From the figure it is clear that,

$$r = |-g| = |-f|$$

$\Rightarrow$ $g = \pm r,\ f = \pm r,$

obviously there are four circles, one in each quadrant as shown in figure.

So the equations of the circles are

$$(x \pm r)^2 + (y \pm r)^2 = r^2$$

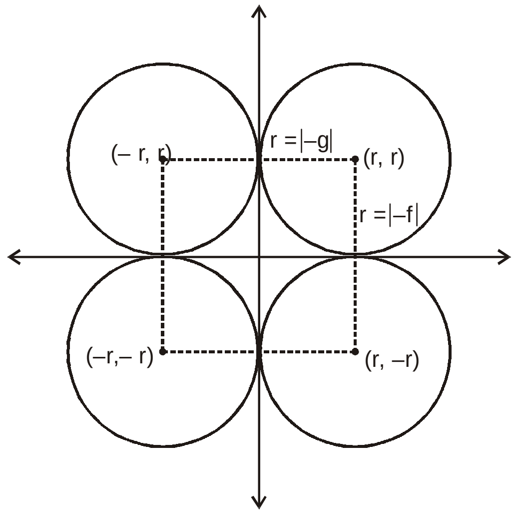


Illustration 6 : ***Find the equation of the circle touching the x-axis at (3, 0) and passing through (1, 2).***

Solution : The abscissa of the centre of the circle is 3; the equation of any circle touching the x-axis at (3, 0) is, by (2),

$$(x - 3)^2 + y^2 + 2fy = 0$$

But (1, 2) lies on the circle ; hence

$$(1 - 3)^2 + 2^2 + 4f = 0 \text{ or } f = -2$$

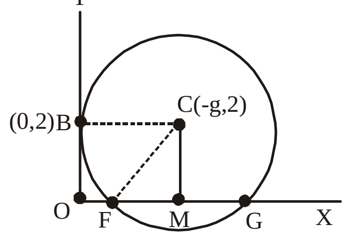
The circle is then

$$(x - 3)^2 + y^2 - 4y = 0 \text{ or } (x - 3)^2 + (y - 2)^2 = 4.$$

Illustration 7 : ***Find the equation of the circle touching the y-axis at (0, 2) and cutting the positive x-axis at F and G such that the length of the chord FG is $4\sqrt{3}$.***

Solution : By (3), the equation of the any circle touching the y-axis at (0, 2) is

$$x^2 + 2gx + (y - 2)^2 = 0 \qquad ....(1)$$



The mid-point of the chord FG is M(fig.);

hence FM = $2\sqrt{3}$. Also, CM is at right angle to FG.

Now, r = CB = CF and CM = 2;

hence, since $CF^2 = FM^2 + MC^2$.

we have $r^2 = 12 + 4 = 16$

But $r^2 = CB^2 = (-g)^2$; hence $-g = \pm 4$. But the abscissa of C is positive, since the abscissa of M is positive; accordingly, $-g = +4$ and the required equation is, from (i),

$$x^2 - 8x + (y - 2)^2 = 0$$

or $$(x - 4)^2 + (y - 2)^2 = 16$$

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EXERCISE - 4

1. Prove that the circle $x^2 + y^2 - 4x - 4y + 4 = 0$ touches the axes at points (2, 0) and (0, 2).
2. Prove that the circle $(x - a)^2 + y^2 = a^2$ touches the y-axis at the origin.
3. Find the equation of the circle which touches the x-axis at (2, 0) and passes through (10, 4)
4. Find the equation of the circle which touches the y-axis at (0, – 3) and intercepts a length 8 on the positive x-axis.
5. Find the equation of the circles which pass through (2, 3) and (–1, 6) and which touch the x-axis.
6. Find the centres and radii of the two circles which pass through (7, 2) and (10, – 1) and which touch the line x = 4.
7. Find the equations of the circles which touch the x-axis at (a, 0) and also touch the line 3x + 4y = 2.
8. Find the centres of the excircles which touch the y-axis, pass through (3, 4) and have their centres on the line 4x + y = 10.
9. Find the equation of circle which touches the axis of x at a distance 3 from the origin and intercepts of length 6 unit on the axis of y.
10. A circle of radius 5 units touches the co-ordinate axes in the first quadrant. If the circle makes one complete roll on x-axis along the positive direction of x-axis, find its equation in the new position.

(E) PARAMETRIC EQUATIONS OF THE CIRCLE

2. The parametric coordinates of any point on the circle $(x - h)^2 + (y - k)^2 = a^2$ are

$x = h + a\cos\theta$

$y = k + a\sin\theta$, where $0 \le \theta < 2\pi$

Note : If h = 0, k = 0, then equation of circle becomes as $x^2 + y^2 = a^2$

its parametric equation are

$x = a\cos\theta$

$y = a\sin\theta$

i.e. any point on the circle $x^2 + y^2 = a^2$ can be taken as $(a\cos\theta, a\sin\theta)$

Illustration 8 : ***Find the parametric equation of a circle having centre at (1, 2) and radius 3.***

Solution : The parametric equation will be

$x = 1 + 3\cos t,\ y = 2 + 3\sin t$

Remark

1. In the above equation, one could have interchanged cosine and sine.
2. If an initial line through the centre (h, k) is assumed, and its point of intersection with the circle is taken as the starting point, then θ measures the angle made by the radius vector of any point with the initial line. Thus two diametrically opposite points can be taken as $(h + a\cos\theta, k + a\sin\theta)$ and $(h + a\cos(\theta + \pi), k + a\sin(\theta + \pi))$

(F) Summary : Equations of the Circle in Various Forms:

1. The simplest equation of the circle is $x^2 + y^2 = r^2$ whose centre is (0, 0) and radius r.
2. The equation $(x - a)^2 + (y - b)^2 = r^2$ represents a circle with centre (a, b) and radius r.

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3. The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ is the general equation of a circle with centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

4. Equation of the circle with points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as extremities of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

5. The equation of the circle through three non-collinear points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

6. Equation of a circle under different conditions :

(i) Touches both the axes with radius a $(x-a)^2 + (y-a)^2 = a^2$

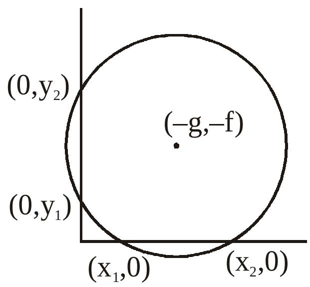
(ii) Touches x-axis only with centre $(x-\alpha)^2 + (y-a)^2 = a^2$ (α, a)

(iii) Touches y-axis only with centre (a, b) $(x-a)^2 + (y-b)^2 = a^2$

(iv) Passes through the origin with centre $x^2 + y^2 - \alpha x - \beta x = 0$

$\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$ and radius $\sqrt{\frac{\alpha^2+\beta^2}{4}}$

3. INTERCEPTS MADE ON AXES



Solving the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with $y = 0$ we get, $x^2 + 2gx + c = 0$ If discriminant $4(g^2 - c)$ is positive, i.e., if $g^2 > c$, the circle will meet the x-axis at two distinct points, say $(x_1, 0)$ and $(x_2, 0)$ where $x_1 + x_2 = -2g$ and $x_1x_2 = c$

The intercept made on x-axis by the circle = $|x_1 - x_2| = 2\sqrt{g^2 - c}$

In the similar manner if $f^2 > c$, intercept made on y-axis = $2\sqrt{f^2 - c}$

If	(i)	$g^2 - c > 0$	$\Rightarrow$	circle cuts the x-axis at two distinct points
	(ii)	$g^2 = c$	$\Rightarrow$	circle touches the x-axis
	(iii)	$g^2 < c$	$\Rightarrow$	circle lies completely above or below the x-axis i.e. it does not intersect x-axis.
If	(i)	$f^2 - c > 0$	$\Rightarrow$	circle cuts the y-axis at two distinct points
	(ii)	$f^2 = c$	$\Rightarrow$	circle touches the y-axis
	(iii)	$f^2 < c$	$\Rightarrow$	circle lies completely on the right side or the left side of the y-axis i.e. it does not intersect y-axis.

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4. THE POSITION OF A POINT WITH RESPECT TO A CIRCLE

Let the equation of the circle be

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, and P(h, k) be any point. Distance between the centre (–g, –f) and P is given by $d = \sqrt{(h+g)^2 + (k+f)^2}$

$\Rightarrow \quad d^2 = (h+g)^2 + (k+f)^2$

$d^2 - r^2 = h^2 + k^2 + 2gh + 2fk + g^2 + f^2 - (g^2 + f^2 - c) = h^2 + k^2 + 2gh + 2fk + c \equiv S_1$

Obviously P lies inside, on or outside the circle as d is less than, equal to or greater than r i.e. $d^2 - r^2$ is negative, zero or positive.

So (i) $S_1(P) < 0 \Rightarrow$ point lies inside the circle.

(ii) $S_1(P) = 0 \Rightarrow$ point lies on the circle.

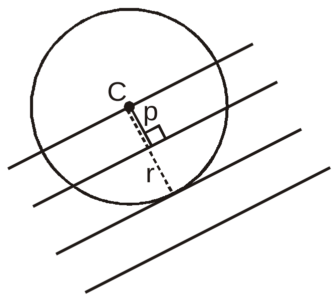
(iii) $S_1(P) > 0 \Rightarrow$ point lies outside the circle.

5. THE POSITION OF A LINE WITH RESPECT TO A CIRCLE

Method I

Let $S = x^2 + y^2 + 2gx + 2fy + c = 0$ be a circle and $L = ax + by + c = 0$ be a line.

Let r be the radius of the circle and p be the length of the perpendicular drawn from the centre (–g, –f) on the line L.Then it can be seen easily from the figure that;



If, (i) $p < r \Rightarrow$ the line intersects the circle in two distinct points

(ii) $p = r \Rightarrow$ the line touches the circle, i.e. the line is a tangent to the circle.

(iii) $p > r \Rightarrow$ the line neither intersects nor touches the circle i.e., passes outside the circle

(iv) $p = 0 \Rightarrow$ the line passes through the centre of the circle.

Method II

Let the line be y = mx + c. Putting y = mx + c in the equation of the circle, you will get a quadratic equation in x. Let D be its discriminant.

If (i) $D > 0 \Rightarrow$ we get two distinct values of x, i.e. the circle will intersect the line in two distinct points.

(ii) $D = 0 \Rightarrow$ we get two equal values of x i.e. the line will touch the circle in one point only i.e. the line will be a tangent to the circle.

(iii) $D < 0 \Rightarrow$ we get no real values of x i.e. the line will not intersect the circle in any point.

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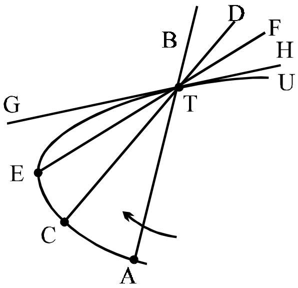
6. DEFINITION OF TANGENT

A tangent to a curve at a point is defined as the limiting position of a secant obtained by joining the given point to another point in the vicinity on the curve as the second point tends to the first point along the curve or as the limiting position of a secant obtained by joining two points on the curve in the vicinity of the given point as both the points tend to the given point

General definition of a tangent to any curve:

In this section we consider the general definition of a tangent to any curve and in the following section we apply the general principles when the curve is a circle.

In fig. T is the point on the curve ACETU at which a tangent is to be drawn.



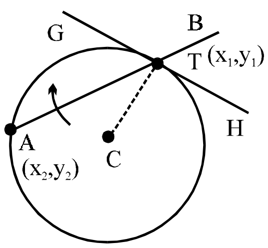
Let ATB, CTD, ETF, ..., be a series of lines passing through T and cutting the curve at the point A, C, E, ..., which are in increasing proximity to T. We can imagine that the lines CTD, ETF, ..., are obtained successively by rotating the line ATB about T in the clockwise direction, the points C, E, ..., creeping closer and closer to T. Eventually we reach a position of the rotating line-indicated by GTH in the figure which meets the curve at T only ; the straight line GTH, obtained in this way, is the tangent to the curve analytically as follows. If (x_1, y_1) are the coordinates of T and (x_2, y_2) are the coordinates of any other point on the curve such as A or C or E, then the tangent at T is the position which the line through (x_1, y_1) and (x_2, y_2) takes up when eventually (x_2, y_2) coincides with (x_1, y_1).

This definition of a tangent is a general definition applicable to any curve.

We add that the normal to a curve at a point T is the perpendicular at T to the tangent at this point.

Application to the circle of the general definition of a tangent:

In fig. T (x_1, y_1) is the point on the circle at which the tangent is to be drawn ; A (x_2, y_2) is any other point on the circle.



(i) Circle, $x^2 + y^2 = r^2$

The equation of the chord AT is

$$(x - x_1)(x_1 + x_2) + (y - y_1)(y_1 + y_2) = 0 \quad ....(1)$$

By the general definition, the tangent GTH is the limiting position of ATB when eventually A coincides with T that is , when $x_2 = x_1$ and $y_2 = y_1$. Accordingly, (1) becomes

$$2(x - x_1)x_1 + 2(y - y_1)y_1 = 0$$

or $$xx_1 + yy_1 = x_1^2 + y_1^2$$

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But, since T lies on the circle, $x_1^2 + y_1^2 = r^2$. Hence the equation of the tangent at T (x_1, y_1) is
$xx_1 + yy_1 = r^2$ ....(2)

Corollary:

The gradient of the tangent is, from (2), $-x_1/y_1$; the gradient of CT, C being the origin is y_1/x_1 ; the product of the gradients is – 1 and hence the tangent at T is at right angles to the radius through T. This can be otherwise stated as : the normal at a point on a circle passes through the centre of the circle.

(ii) Circle in general form

The equation of the chord AT is,

$$(x - x_1)(x_1 + x_2 + 2g) + (y - y_1)(y_1 + y_2 + 2f) = 0 \quad ......(3)$$

and the tangent GTH is the limiting position of ATB when eventually A coincides with T that is, when $x_2 = x_1$ and $y_2 = y_1$. Accordingly, (3) becomes

$$(x - x_1)(x_1 + g) + (y - y_1)(y_1 + f) = 0$$

or $$x(x_1 + g) + y(y_1 + f) = x_1^2 + y_1^2 + gx_1 + fy_1. \quad ......(4)$$

But, since T is on the circle,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad ....(5)$$

Hence, by equation (4) and (5),

$$x(x_1 + g) + gx_1 + y(y_1 + f) + fy_1 + c = 0 \quad ....(6)$$

This is the equation of the tangent at T (x_1, y_1) on the circle in general form.

6.1 CONDITION THAT y = mx + c SHOULD BE A TANGENT TO THE CIRCLE

The required condition will be derived, first, from the theorem that the radius to the point of contact, P, is perpendicular to the tangent at P or in other words, the length of the perpendicular from the centre of the circle to the tangent is equal to the radius and, second, from the Euclidean definition of a tangent.

Method I

(i) We first take the circle to be $x^2 + y^2 = r^2$.

If p is the length of the perpendicular from O (the origin) to the line $mx - y + c = 0$, then,

$$r \equiv p = \pm \frac{c}{\sqrt{(m^2+1)}} \text{ ; hence}$$

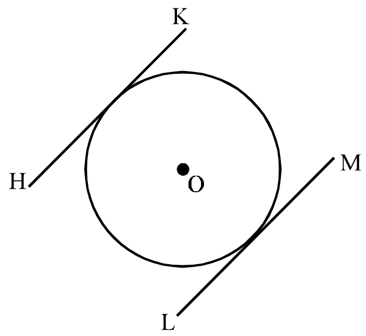
$$c = \pm r\sqrt{(1+m^2)} \quad ......(1)$$

This is the required condition.

There are, thus, two lines of gradient m which touch the circle ; they are

$$y = mx + r\sqrt{(1+m^2)},\ y = mx - r\sqrt{(1+m^2)} \quad ......(2)$$

and are represented by HK and LM in fig.



For the circle in general form, the centre is (– g, – f) ; then

$$p \equiv r = \pm \frac{(-gm + f + c)}{\sqrt{(1+m^2)}}$$

and the required condition is then

$$c = mg - f \pm r\sqrt{(1+m^2)} \quad ....(3)$$

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where r = $\sqrt{(g^2 + f^2 - c)}$.

Method II

In this method we start from the definition of a tangent, namely, that if y = mx + c is a tangent, then it meets the circle in one point. Substitute (mx + c) for y in the equation of the circle $x^2 + y^2 = r^2$; then we obtain the following quadratic equation in x

$$x^2(1 + m^2) + 2mcx + c^2 - r^2 = 0. \qquad ....(4)$$

The roots of this equation are, in general, the abscissae of the two points of intersection of the line with the circle; if the line is a tangent, the two roots must be equal, the condition for which is

$$(mc)^2 - (1 + m^2)(c^2 - r^2) = 0$$

which reduces to

$$c^2 = r^2(1 + m^2) \text{ or } c = \pm r\sqrt{(1+m^2)}, \text{ as in (1).}$$

When the equation of the circle is given in general form, substitute (mx + c) for y in the equation of the circle and apply the condition for equal roots in the resulting quadratic equation in x.

It is sometimes convenient to say that, if a line is a tangent to a circle, it meets the circle at "two coincident points", thus drawing attention to the fact that the two roots of (4), for example, are equal; this description, however, is an analytical fiction, for it is the essence of tangency that a tangent meets the circle at one point only.

Illustration 9 : ***Find the equation of the circle whose centre is (3, 4) and which touches the line 5x + 12y = 1.***

Solution : Let r be the radius of the circle. Then r = distance of the centre i.e. point (3, 4) from the line 5x + 12y – 1 = 0

i.e. $$r = \left|\frac{15+48-1}{\sqrt{25+144}}\right| = \frac{62}{13}$$

Hence the equation of the required circle is $(x-3)^2 + (y-4)^2 = \left(\frac{62}{13}\right)^2$

Illustration 10 : **Find the equation of the circle which has its centre at the point (6, 1) and touches the straight line 5x + 12y = 3.**

Solution : The radius of the circle is perpendicular distances from the centre to the straight line 5x + 12y = 3.

$$R = \frac{5(6)+12(1)-3}{\sqrt{5^2+12^2}} = 3.$$

The equation of the circle is $(x-6)^2 + (y-1)^2 = 9$

i.e. $x^2 + y^2 - 12x - 2y + 28 = 0$

The coordinates of the point of contact of the tangents to a Circle:

The tangents to the circle $x^2 + y^2 = r^2$, of given gradient m, are

$$y = mx \pm r\sqrt{(1+m^2)} \qquad ....(1)$$

The normal to the tangents is $y = -\frac{1}{m}x$ or

$x = -my$ ....(2)

The coordinates of the points of contact are given by solving (1) and (2).

by means of (2), (1) becomes

$$y(1+m^2) = \pm r\sqrt{(1+m^2)}$$

or $$y = \pm\frac{r}{\sqrt{(1+m^2)}}$$

Then, by (2),

$$x = \mp\frac{mr}{\sqrt{(1+m^2)}}.$$

If M denotes $\sqrt{(1+m^2)}$, the coordinates of the points of contact are

$$\left(-\frac{mr}{M}, \frac{r}{M}\right) \text{ and } \left(\frac{mr}{M}, -\frac{r}{M}\right) \quad ......(3)$$

These formulae need not be remembered; in numerical examples, including the equations of circles in general form, the general principles should be applied, as illustrated in Illustrations 11 and 12 below.

Illustration 11 : ***Find the tangents of gradient 3/4 to the circle $x^2 + y^2 = 9$ and the coordinates of the point of contact.***

Solution : The radius of the circle is 3. The equations of the tangents are, of the form

$$y = \frac{3}{4}x + c$$

wherec $= \pm r\sqrt{(1+m^2)} = \pm 3\sqrt{\left[1+\left(\frac{3}{4}\right)^2\right]} = \pm\frac{15}{4}$.

The tangents are then :

$3x - 4y + 15 = 0$, $3x - 4y - 15 = 0$.

The gradient of the normal is – 4/3 and the equation of the normal is

$y = -\frac{4}{3}x$ or $4x + 3y = 0$.

The coordinates of the two points of contact are obtained by solving

(i) $4x + 3y = 0$ and $3x - 4y + 15 = 0$, and

(ii) $4x + 3y = 0$ and $3x - 4y - 15 = 0$.

The results are : (i) $\left(-\frac{9}{5}, \frac{12}{5}\right)$ and (ii) $\left(\frac{9}{5}, -\frac{12}{5}\right)$.

Illustration 12. Find the tangents of gradient 4/3, to the circle

$$\mathbf{x^2 + y^2 - 10x + 24y + 69 = 0}$$

and the coordinates of the points of contact.

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Solution: The given circle is

$$(x-5)^2 + (y+12)^2 = 100;$$

its centre, C, is (5, – 12) and its radius is 10.

Let $y = mx + c$ or $mx - y + c = 0$ be the equation of a tangent. If p is the length of the perpendicular from C to the tangent it is given by

$$p = \pm \frac{5m + 12 + c}{\sqrt{(1+m^2)}}$$

Now, p = r = 10; hence

$$(5m + 12 + c)^2 = 100\,(1 + m^2)$$

from which, with m = 4/3, 3c + 56 = ± 50; thus c = – 2 or c = – 106/3.

The tangents are

$$y = \frac{4}{3}x - 2 \text{ and } y = \frac{4}{3}x - \frac{106}{3}$$

or $\quad 4x - 3y - 6 = 0$ and $4x - 3y - 106 = 0$ ....(i)

The normal at either point of contact is of gradient –3/4 and since it passes through C(5, – 12) its equation is

$$y + 12 = -\frac{3}{4}(x-5)$$

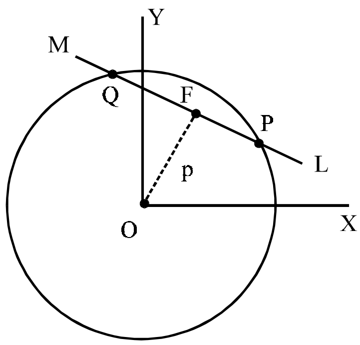
$3x + 4y + 33 = 0$ ....(ii)

The coordinates of the points of contact are then found by solving (ii) with each of (i) in turn ; the results are : (– 3, – 6) and (13, – 18).

6.2 CONDITION THAT A GIVEN LINE SHOULD CUT A CIRCLE

(i) Circle, $x^2 + y^2 = r^2$

In fig. LM is a straight line cutting the circle at P and Q; *p* is the length of the perpendicular from O to LM. The geometrical condition that LM should cut the circle in two points is : p < r.



Let y = mx + c be the equation of LM in gradient-form ; then,

$$p = \pm \frac{c}{\sqrt{(1+m^2)}}$$

or $\quad p^2 = c^2/(1 + m^2)$,

whatever the sign of c. The required condition is then

$c^2 < (1 + m^2)r^2$ ......(1)

(ii) Circle in general form

Here, the centre is (–g, – f) and $r^2 = g^2 + f^2 - c$; the line being written as $mx - y + c = 0$, then p is given by

$$p = \pm \frac{-gm + f + c}{\sqrt{(1+m^2)}}$$

The condition is then

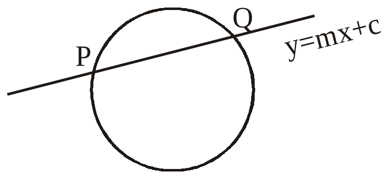
$(gm - f - c)^2 < (1 + m^2)(g^2 + f^2 - c)$ ....(3)

In numerical examples the conditions (1) and (3) should not be quoted but should be derived from the calculated values of p and r.

(A) The coordinates of the points at which a given line cuts a circle:

(i) Circle, $x^2 + y^2 = r^2$

The coordinates of P and Q in fig. satisfy the equation of the circle and the equation of the line which we take to be in the gradient-form ; the coordinates are given by solving the equations



$x^2 + y^2 = r^2$ and $y = mx + c$

Substitute the expression for $y(\equiv mx + c)$ in the first equation ;

then $x^2(1 + m^2) + 2mcx + c^2 - r^2 = 0$

This is a quadratic equation in x the roots of which (say, x_1 and x_2) are the abscissae of P and Q. The corresponding ordinates (y_1 and y_2) are given by

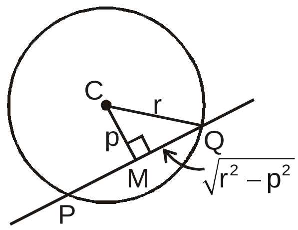
$y_1 = mx_1 + c$ and $y_2 = mx_2 + c$

In some numerical examples, it may be advantageous to substitute $x = (y - c)/m$ in the equation of the circle and so to derive, first, the values of y_1 and y_2.

(ii) If the circle is not centred at the origin, a similar procedure is carried out, as illustrated in Illustration 14 below.

(B) Length of the chord intercepted by a line on the circle:

Method I



Let p be length of perpendicular from centre to the line L = 0

It can be easily seen from the figure that

Length of the chord = PQ = 2MQ = $2\sqrt{r^2 - p^2}$

Method II

Let the line be y = mx + c. Put y = mx + c in the equation of the circle so that you get a quadratic equation in x. Let x_1, x_2 be its roots.

Then $(x_1 + x_2)$ and x_1x_2 can be easily found and thereby $(x_1 - x_2)$ can be found by the following formula

$(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$.

We know that both points of intersection satisfy $y_1 = mx_1 + c$, and $y_2 = mx_2 + c$

So, $y_1 - y_2 = m(x_1 - x_2)$

The length of the intercept is then $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$$= \sqrt{(x_1 - x_2)^2 + m^2(x_1 - x_2)^2}$$

$$= |x_1 - x_2|\sqrt{1 + m^2}$$

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Tip for Solving Questions

The abovesaid Method II is easier than finding (x_1, y_1) & (x_2, y_2) and then finding length of chord by the formula $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

Illustration 13 : ***Prove that $x + y - 1 = 0$ cuts the circle $x^2 + y^2 = 25$ at two points and find their coordinates.***

Solution : The perpendicular to the line $-x - y + 1 = 0$ from the origin is $\frac{1}{\sqrt{(1+1)}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$; this is clearly less than the radius, 5; the line cuts the circle in two points.

Write the equation of the line as : $y = 1- x$ and substitute this expression for y in the equation of the circle ; then

$$x^2 + (1 - x)^2 = 25 \text{ or } x^2 - x - 12 = 0$$

or $\quad (x - 4)(x + 3) = 0$

The roots are 4 and – 3; these are the abscissae of the points of intersection. The corresponding ordinates are obtained by substituting these abscissae successively in the equation $y = 1 - x$, the results being – 3 and 4. The coordinates of the points of intersection are (4, – 3) and (– 3, 4).

Illustration 14 : ***Prove that the line $2x - 3y + 30 = 0$ cuts the circle $x^2 + y^2 - 2x - 4y - 164 = 0$ at two points and find their coordinates.***

Solution : The circle is : $(x - 1)^2 + (y - 2)^2 = 169$; its centre, C, is (1, 2) and its radius is 13.

The length of the perpendicular from C to the line is given by

$$p = \frac{2(1)-3(2)+30}{\sqrt{(4+9)}} = \frac{26}{\sqrt{13}} = 2\sqrt{13}\,.$$

Thus, p is less than the radius 13 the line cuts the circle in two points.

Write the equation of the line as : $x = \frac{1}{2}(3y - 30)$ and substitute this expression for x in the equation of the circle; then

$$\frac{1}{4}(3y - 30)^2 + y^2 - (3y - 30) - 4y - 164 = 0$$

which reduces to

$$y^2 - 16y + 28 = 0 \text{ or } (y - 2)(y - 14) = 0.$$

The roots are : $y = 2$ and $y = 14$; substitute these in turn in the equation $2x - 3y + 30 = 0$; the corresponding values of x are – 12 and 6. The points of intersection are (– 12, 2) and (6, 14).

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EXERCISE - 5

1. Prove that the line $x + y + 1 = 0$ and circle $x^2 + y^2 - 2x + 8y + 13 = 0$ intersect, and find the points of intersection.
2. Prove that the tangents to circle $(x - a)^2 + (y - b)^2 = r^2$ at its points of intersection with line $x - a = (y - b) \tan a$ are $(x - a) \sin a + (y - b) \cos a = \pm r$.
3. The line $ax + by = 1$ cuts the circle $c(x^2 + y^2) = 1$, $(c > 0)$, at two points A and B such that OA is perpendicular to OB ; prove that $2c = a^2 + b^2$. Prove also that if $ax + ky = 1$ is a tangent to the circle then $k^2 = c - a^2$.
4. The line of gradient $m = 1$ through the point $A\ (2\sqrt{2}, -4\sqrt{2})$ cuts the circle $x^2 + y^2 = 37$at B and C; find the length of the chord BC.
5. The line $2x + 4y = 15$ cuts the $x^2 + y^2 - 2x - 4y + 1 = 0$ at A and B. Prove that the foot of the perpendicular from the origin to AB is the mid-point of AB.

6.3 EQUATION OF TANGENT TO A CIRCLE AT (X_1, Y_1) IN THE FORM T = 0

Let $P(x_1, y_1)$ be any point lying on the circle. If we replace x^2 by xx_1, y^2 by yy_1, x by $\frac{1}{2}(x + x_1)$ and y by $\frac{1}{2}(y + y_1)$ in the equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, we obtain $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$. This is also called T. So equation of the tangent is T = 0

Equation of the circle	Equation of the tangent at $P(x_1, y_1)$
1. $x^2 + y^2 = r^2$	$T \equiv xx_1 + yy_1 - r^2 = 0$
2. $(x - h)^2 + (y - k)^2 = r^2$	$T \equiv (x - h)(x_1 - h) + (y - k)(y_1 - k) - r^2 = 0$
3. $x^2 + y^2 + 2gx + 2fy + c = 0$	$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Proof :

1. Let (x_2, y_2) be a point on the circle $x^2 + y^2 = a^2$. We have $x_1^2 + y_1^2 = a^2$ and $x_2^2 + y_2^2 = a^2$.
   Hence $x_2^2 - x_1^2 = -(y_2^2 - y_1^2)$

   or $$-\frac{x_2 + x_1}{y_2 + y_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

   Now if (x_2, y_2) tends to (x_1, y_1). The chord becomes tangent to the circle at (x_1, y_1) and for the slope then we get $\frac{y_2 - y_1}{x_2 - x_1} = \frac{-2x_1}{2y_1} = -\frac{x_1}{y_1}$.

   Hence, in the limiting position the equation of the line becomes

   $$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$

   or $xx_1 + yy_1 = x_1^2 + y_1^2 = a^2$

   or $xx_1 + yy_1 = a^2$

   which is the required equation of the tangent.

The slope of the tangent is $-\dfrac{x_1}{y_1}$ and that the line joining (x_1, y_1) to the centre is $\dfrac{y_1}{x_1}$. Since their product is – 1, they are perpendicular.

2. Put $(x-h) = X$, $(y-k) = Y$

So $(x-h)^2 + (y-k)^2 = r^2$ becomes $X^2 + Y^2 = r^2$, the tangent to which is $XX_1 + YY_1 = r^2$

Since $X = (x-h)$, so $X_1 = (x_1 - h)$

and $Y = (y-k)$, so $Y_1 = (y_1 - k)$

So equation of tangent becomes $(x-h)(x_1-h) + (y-k)(y_1-k) - r^2 = 0$.

3. Let (x_2, y_2) be a point near (x_1, y_1) on the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$.

The equation of the straight line joining (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Since the point (x_1, y_1) and (x_2, y_2) lie on the circle, we have

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$

Hence $x_2^2 - x_1^2 + y_2^2 - y_1^2 + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0$

or $\dfrac{y_2 - y_1}{x_2 - x_1} = -\dfrac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}$

Now, if (x_2, y_2) tends to (x_1, y_1) along the circle, we get $\dfrac{y_2 - y_1}{x_2 - x_1} = -\dfrac{x_1 + g}{y_1 + f}$

Therefore the equation of the tangent is $y - y_1 = -\dfrac{x_1 + g}{y_1 + f}(x - x_1)$

or $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) - (x_1^2 + y_1^2 + 2gx_1 + 2fy_1) = 0$

or $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

The slope of the tangent is $-\dfrac{x_1 + g}{y_1 + f}$ and the line joining (x_1, y_1) to the centre $(-g, -f)$ is $\dfrac{y_1 + f}{x_1 + g}$.

Since their product is – 1, they are perpendicular to each other. Hence the result.

For a given point (x_1, y_1) and a given circle, two tangents, real or imaginary, can be drawn to pass through (x_1, y_1). In other words two tangents, real or imaginary, can be drawn from a point in the plane of the circle. The tangents are real and distinct if the point is outside the circle, the tangents are real and coincident if the point is on the circle and the tangents are imaginary if the point is inside the circle.

The points where the tangents touch the given circle are called points of contact and the line joining the points of contact is called the chord of contact.

Equation of Tangent to the Circle $x^2 + y^2 = a^2$ at the point $(a\cos\alpha, a\sin\alpha)$

Since the equation of tangent to the circle $x^2 + y^2 = a^2$ at the point (x_1, y_1) is $xx_1 + yy_1 = a^2$, so to get the equation of tangent at $(a\cos\alpha, a\sin\alpha)$, put $x_1 = a\cos\alpha$, $y_1 = a\sin\alpha$ in

$xx_1 + yy_1 = r^2$ to get

$x\cos\alpha + y\sin\alpha = a$

Tip for Solving Questions

The point of intersection of the tangents at the points (a cosα, a sinα) and (a cosβ, a sinβ) is

$$\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{a\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

Illustration 15 : ***Find the equations of the tangent at A(4, – 2) on the circle $x^2 + y^2 = 20$.***

Solution : The tangent at A is

$$x(4) + y(-2) = 20 \text{ or } 2x - y = 10$$

Illustration 16 : ***Find the equations of the tangent at A(2, 3) to the circle $x^2 + y^2 + 6x - 4y - 13 = 0$.***

Solution : Here, g = 3, f = – 2 and c = – 13, the tangent is

$x.2 + y.3 + 3(x + 2) - 2(y + 3) - 13 = 0$

or $5x + y - 13 = 0$

EXERCISE - 6

1. Find the equation of the tangent at the point (4, 3) on the circle $x^2 + y^2 = 25$.
2. Find the equation of the tangent at the origin to the circle $x^2 + y^2 - 4x - 10y = 0$.
3. Prove that the tangents to the circle $x^2 + y^2 = 169$ at the points (5, 12), (–5, 12) and (5, – 12) form a rhombus whose diagonals are the x and y axes and whose area is $\frac{1}{130}(169)^2$.
4. The circle $(x - 2)^2 + (y - 2)^2 = r^2$ cuts the y-axis at A and B ; if the tangents at A and B are perpendicular, prove that $r = 2\sqrt{2}$.
5. The tangents at A(5, – 4) and B(4, 5) on the circle $x^2 + y^2 = 41$ intersect at C. Find the coordinates of C and prove that OACB is a square of side $\sqrt{41}$.
6. The circle $x^2 + y^2 - kx + ky = 0$ cuts the x-axis at A and B; prove that the tangents at A and B are x = y and x + y = k.
7. Prove that the tangets at the points of intersection of the line 3x + y = 12 and the circle $x^2 + y^2 - 6x + 4y + 8 = 0$ are perpendicular.
8. Find the equation of the circle which touches x + y = 3 at (1, 2) and which passes through (2, 2).

6.4 EQUATION OF THE TANGENT IN TERMS OF SLOPE m

There will be two parallel tangents of slope m to any circle

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Circle		Tangents
1.	$x^2 + y^2 = r^2$	$y = mx \pm r\sqrt{(1+m^2)}$
2.	$(x-h)^2 + (y-k)^2 = r^2$	$y - k = m(x-h) \pm r\sqrt{(1+m^2)}$
3.	$x^2 + y^2 + 2gx + 2fy + c = 0$	$y + f = m(x+g) \pm \sqrt{(g^2+f^2-c)}\ \sqrt{(1+m^2)}$

Proof :

1. Proof to first equation is already given in article 6.1, equation (2)
2. Put $(x-h) = X$and $(y-k) = Y$, so $(x-h)^2 + (y-k)^2 = r^2$ reduces to $X^2 + Y^2 = r^2$ the tangents to which are $Y = mX \pm r\sqrt{(1+m^2)}$

   Putting back the values of X and Y, we get equation of tangents as $y - k = m(x-h) \pm r\sqrt{(1+m^2)}$
3. $x^2 + y^2 + 2gx + 2fy + c = 0$ can be reduced to $(x+g)^2 + (y+f)^2 = g^2 + f^2 - c = r^2$. So equation of tangents becomes $y + f = m(x+g) \pm \sqrt{(g^2+f^2-c)}\ \sqrt{(1+m^2)}$.

Tip for Solving Questions

The points of contact of the tangents $y = mx + a\sqrt{1+m^2}$ and $y = mx - a\sqrt{1+m^2}$ on the circle $x^2 + y^2 = a^2$ are $\left(\frac{-ma}{\sqrt{1+m^2}}, \frac{a}{\sqrt{1+m^2}}\right)$ and $\left(\frac{ma}{\sqrt{1+m^2}}, \frac{-a}{\sqrt{1+m^2}}\right)$ respectively.

6.5 NUMBER OF TANGENTS TO A CIRCLE FROM A POINT

	POSITION OF POINT	NO. OF TANGENTS
1.	the point lies outside the circle	two real & distinct tangents can be drawn
2.	the point lies on the circle	only one real tangent can be drawn
3.	the point lies inside the circle	no real tangent can be drawn

Proof :

By choosing the origin at the centre of the circle, we can take the equation of the circle as $x^2 + y^2 = a^2$.

Since $y = mx + a\sqrt{1+m^2}$ is always tangent to the circle the tangents that pass through the point (x_1, y_1) must also be of the above form and then we get

$y_1 = mx_1 + a\sqrt{1+m^2}$

or $a^2(1 + m^2) = (y_1 - mx_1)^2$

or $a^2(1 + m^2) = y_1^2 - 2x_1y_1m + m^2x_1^2$

or $(a^2 - x_1^2)m^2 + 2x_1y_1m + (a^2 - y_1^2) = 0$

which is a quadratic in m and hence gives two values of m determining two tangents. If the roots are real then the tangents are real and if the roots are imaginary, the tangents are imaginary.

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Thus the tangents are real if $x_1^2y_1^2 - (a^2 - x_1^2)(a^2 - y_1^2) \geq 0$

i.e., if $x_1^2 + y_1^2 - a^2 \geq 0$

i.e., if (x_1, y_1) lies outside or on the circle

Similarly, the tangents are imaginary if $x_1^2 + y_1^2 < a^2$, i.e., if (x_1, y_1) lies inside the circle.

6.6 EQUATIONS OF THE TANGENTS FROM AN EXTERNAL POINT TO A CIRCLE

Method I

(i) Circle, $x^2 + y^2 = r^2$.

Let $A(x_1, y_1)$ be the given external point. The line $y = mx + b$ is a tangent if

$$b^2 = r^2 (1 + m^2);$$

also, if the line passes through A, then

$$b = y_1 - mx_1.$$

It follows that $(y_1 - mx_1)^2 = r^2(1 + m^2)$

or $\quad m^2(x_1^2 - r^2) - 2mx_1y_1 + y_1^2 - r^2 = 0 \qquad$ ....(1)

This is a quadratic equation in m; the two roots are, say, m_1 and m_2. Thus there are two tangents to the circle from A; their equations are

$y - y_1 = m_1(x - x_1) \qquad$ ....(2)

and $\quad y - y_1 = m_2(x - x_1) \qquad$ ....(3)

The roots of (1) are real and distinct if

$$(-x_1y_1)^2 > (x_1^2 - r^2)(y_1^2 - r^2)$$

that is, if

$$x_1^2 + y_1^2 > r^2$$

But, O being the origin, $OA^2 = x_1^2 + y_1^2$; hence the condition for real and distinct values of the roots of (1) is : $OA^2 > r^2$ - in other words A is outside the circle.

The condition for equal roots is : $x_1^2 + y_1^2 = r^2$, that is, A must lie on the circle; in this event there is one and only one tangent through A.

The roots are imaginary if $x_1^2 + y_1^2 < r^2$; hence A lies within the circle.

All these qualitative results are evident from simple geometrical concepts.

The normal to the tangent whose equation is given by (2) is

$$y = -\frac{1}{m_1}x \qquad \text{or} \qquad x + m_1y = 0 \qquad ....(4)$$

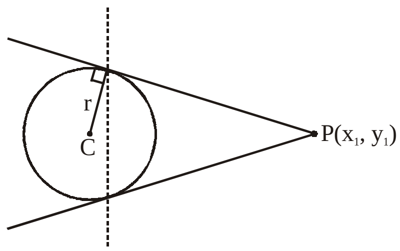
The coordinates of the point of contact are then found by solving (2) and (4).

Similarly, the coordinates of the point of contact of the second tangent are found by solving (3) and the equation $x + m_2y = 0$

(ii) Circle in general form.

The general procedure is similar to that in (i) and is best described in terms of Illustrations below.

Method II



Let $\quad S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be the circle,

$$T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

and $\quad S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

then $\;T^2 = SS_1$ represents the combined equation of the tangents from a point.

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Tip for Solving Questions

Method II is generally a very time consuming method, so it is very rarely used in solving questions. Generally students should prefer Method I.

Illustration 17 : ***Find the pair of tangents from A(12, – 2) to the circle $x^2 + y^2 = 74$.***

Solution : Here, $OA^2 = 144 + 4 = 148$ and $OA^2 > r^2$; accordingly, A is external to the circle and there are two distinct tangents.

The line $y = mx + b$ is a tangent if $b^2 = 74(1 + m^2)$; since A on the line, then $-2 = 12m + b$ or $b = -2(1 + 6m)$. Hence m is given by

$$4(1 + 12m + 36m^2) = 74(1 + m^2)$$

or $$35m^2 + 24m - 35 = 0$$

or $$(7m - 5)(5m + 7) = 0$$

The gradients of the tangent are 5/7 and –7/5 and their equations are

$$y + 2 = \frac{5}{7}(x - 12) \quad \text{and} \quad y + 2 = -\frac{7}{5}(x - 12)$$

or $5x - 7y - 74 = 0$ and $7x + 5y - 74 = 0$

The normal to the first tangent is $7x + 5y = 0$ and the coordinates of the point of contact are obtained by solving

$$5x - 7y - 74 = 0 \text{ and } 7x + 5y = 0.$$

The result is : (5, – 7).

Similarly, the point of contact of the second tangent is (7, 5).

Illustration 18 : ***Find the pair of tangents from A (5, 10) to the circle $x^2 + y^2 + 4x - 2y - 8 = 0$.***

Solution : The circle is : $(x + 2)^2 + (y - 1)^2 = 13$; the centre C is (–2, 1) and the radius is $\sqrt{13}$.

Lety = mx + b be a tangent ; then, since it passes through A,

$$b = 10 - 5m \qquad ....(1)$$

If p is the length of the perpendicular from C to the tangent, then $p = r = \sqrt{13}$. But, the equation of the tangent being written as $mx - y + b = 0$, p is given by

$$p = \pm\frac{-2m - 1 + b}{\sqrt{(1+m^2)}} \qquad ....(2)$$

or, by means of (1)

$$p = \pm\frac{9 - 7m}{\sqrt{(1+m^2)}}.$$

Hence, m is given by

$$(9 - 7m)^2 = 13(1 + m^2)$$

or $$18m^2 - 63m + 34 = 0$$

or $$(3m - 2)(6m - 17) = 0$$

The gradients are 2/3 and 17/6; the tangents are

$$y - 10 = \frac{2}{3}(x - 5)$$

and $$y - 10 = \frac{17}{6}(x - 5)$$

or $$2x - 3y + 20 = 0$$

and $$17x - 6y - 25 = 0 \qquad ....(3)$$

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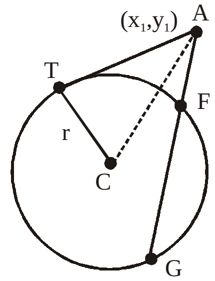
EXERCISE - 7

1. Find the equations of the tangents from (5, 4) to the circle $x^2 + y^2 - 6x - 4y + 9 = 0$. Also find the points of contact.
2. Prove that the two tangents from the origin to the circle in general form, O being external to the circle, are the line-pair $c(x^2 + y^2) - (gx + fy)^2 = 0$, $(c > 0)$.
3. Tangents PA and PB are drawn from P (x_1, y_1) to the circle $x^2 + y^2 = r^2$. Using the parametric equation $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$, prove that the tangent of one of the angles between PA and PB is $(t_1 - t_2)/(1 + t_1t_2)$ where t_1, t_2 are the roots of the equation
$t^2(x_1^2 - r^2) - 2x_1y_1t + y_1^2 - r^2 = 0$.

6.7 LENGTH OF A TANGENT FROM AN EXTERNAL POINT



In figure, C is the centre of a circle, $A(x_1, y_1)$ is an external point and AT is a tangent from A, T being the point of contact. The length of AT is given by

$$AT^2 = CA^2 - CT^2 = CA^2 - r^2 \qquad ......(1)$$

where r is the radius.

(i) Circle, $x^2 + y^2 = r^2$.
C is (0, 0) and we have at once from (1)

$$AT^2 = x_1^2 + y_1^2 - r^2 = S_1 \Rightarrow \text{length of tangent} = \sqrt{S_1} \qquad ....(2)$$

(ii) Circle in general form.
C is $(-g, -f)$ and $r^2 = g^2 + f^2 - c$; also $CA^2 = (x_1 + g)^2 + (y_1 + f)^2$. Hence, by (1),

$$AT^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = S_1$$

$$\Rightarrow \quad \text{length of tangent} = \sqrt{S_1} \qquad ....(3)$$

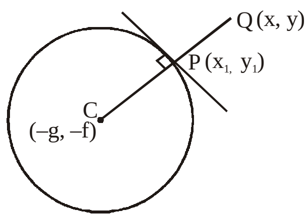
Note :

The lengths of both the tangents drawn from a point to a circle are equal.

EXERCISE - 8

1. Find the lengths of the tangents from the given points to the given circles -
(i) (3, 6), $x^2 + y^2 = 25$ (ii) (–2, 3), $x^2 + y^2 + 2x + 5y = 0$

7. DEFINITION OF NORMAL



The normal to a circle at a point is defined as the straight line passing through the point and perpendicular to the tangent at that point. ***Clearly every normal passes through the centre of the circle***.

The equation of the normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$

at any point (x_1, y_1) lying on the circle is $\dfrac{y_1 + f}{x_1 + g} = \dfrac{y - y_1}{x - x_1}$

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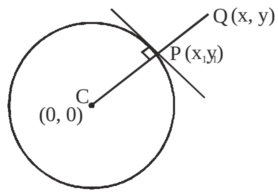
In particular, equation of the Normal to the circle

$x^2 + y^2 = a^2$ at (x_1, y_1) is $\frac{y}{x} = \frac{y_1}{x_1}$.



8. DEFINITION OF CHORD OF CONTACT

From a point $P(x_1, y_1)$ two tangents PA and PB can be drawn to the circle. The chord AB joining the points of contact A and B of the tangents from P is called the chord of contact of $P(x_1, y_1)$ with respect to the circle. Its equation is given by T = 0.

8.1 CHORD OF CONTACT OF TANGENTS FROM AN EXTERNAL POINT

(i) Circle $x^2 + y^2 = r^2$.

In fig. let T and U be the points of contact of tangents from the external point $A(x_1, y_1)$. The chord TU is called the chord of contact with reference to A.

Let T be (h, k) ; then the tangent at T is,

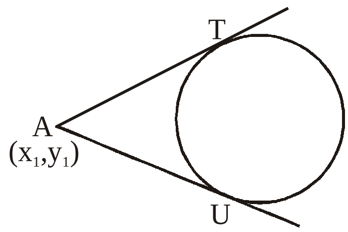
$xh + yk = r^2$ ....(1)

Similarly, if U is (H, K), the tangent at U is

$xH + yK = r^2$ ....(2)

Since A (x_1, y_1) is on both the lines (1) and (2) then

$x_1h + y_1k = r^2$ ....(3)



and $x_1H + y_1K = r^2$ ....(4)

But (3) is the condition that T (h, k) lies on the line

$xx_1 + yy_1 = r^2$ ....(5)

Similarly, (4) is the condition that U(H, K) lies on the line (5). Thus both T and U lie on this line and, consequently, the chord of contact is

$xx_1 + yy_1 = r^2$ ....(6)

Clearly, the line (6) is perpendicular to the line joining A to the centre of the circle.

Note :

The equation (6) has the same appearance as the equation of the tangent for which the point of contact is (x_1, y_1).

(ii) Circle in general form.

By arguments similar to those in (i) it is easily found that the chord of contact is

$x(x_1 + g) + y (y_1 + f) + gx_1 + fy_1 + c = 0$ ....(7)

Alternatively, Change the origin to the centre (–g, –f) and, let (u, v) be coordinates corresponding to (x, y). Then

$x = u – g,\ y = v – f\ ;\ x_1 = u_1 – g,\ y_1 = v_1 – f$ ....(8)

The equation of the circle is

$(x + g)^2 + (y + f)^2 = g^2 + f^2 – c = r^2$

or, with reference to the new axes,

$u^2 + v^2 = r^2$

By (6) the chord of contact is $uu_1 + vv_1 = r^2$ or, by (8),

$$(x + g)(x_1 + g) + (y + f)(y_1 + f) = g^2 + f^2 - c$$

from which (7) follows.

8.2 EQUATION OF CHORD OF CONTACT

The equations of the Chords of contact of the tangents drawn from the point (x_1, y_1) to the different circles are:

	Circle	Chord of Contact
(i)	$x^2 + y^2 = a^2$	$xx_1 + yy_1 = a^2$
(ii)	$x^2 + y^2 + 2gx + 2fy + c = 0$	$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

Illustration 21 : ***Find the co-ordinates of the point from which tangents are drawn to the circle $x^2 + y^2 - 6x - 4y + 3 = 0$ such that the mid point of its chord of contact is (1, 1).***

Solution : Let the required point be $P(x_1, y_1)$. The equation of the chord of contact of P with respect to the given circle is

$$xx_1 + yy_1 - 3(x + x_1) - 2(y + y_1) + 3 = 0$$

The equation of the chord with mid-point (1, 1) is

$$x + y - 3(x + 1) - 2(y + 1) + 3 = 2 - 6 - 4 + 3 \Rightarrow 2x + y = 3$$

Equating the ratios of the coefficients of x, y and the constant terms and solving for x, y we get

$$x_1 = -1,\ y_1 = 0.$$

Illustration 22: ***Find the length of the chord of contact, CD of tangents from (3, 4) to the circle $x^2 + y^2 = 4$.***

Solution : The chord of contact is $3x + 4y = 4$. If M is the mid-point of CD, then OM =

$$\frac{4}{\sqrt{(3^2 + 4^2)}} = \frac{4}{5} \text{ and } CM^2 \equiv r^2 - OM^2 = 4 - \frac{16}{25} = \frac{84}{25}.$$

Hence $CD = 2.\sqrt{\frac{84}{25}} = \frac{4}{5}\sqrt{21}$.

EXERCISE - 9

1. Find the equations of the chords of contact from the given points to the given circles -

   (i) (5, 3), $x^2 + y^2 = 14$ (ii) (4, – 3), $x^2 + y^2 - 2x + 4y + 3 = 0$

2. A (– 12, – 6) lies on the chord of contact of tangents from B(3, – 8) to the circle $x^2 + y^2 - 4x - 2y = 44$. Verify that B lies on the chord of contacts from A.

3. P(6, – 7) and Q(a, 5) are points each of which lies on the chord of contact of tangents from the other to the circle $x^2 + y^2 - 3x + 5y + 4 = 0$; find the value of a.

4. Prove that the chord of contact of tangents from (h, k) to the circle $x^2 + y^2 = r^2$ subtends a right angle at the origin if $h^2 + k^2 = 2r^2$.

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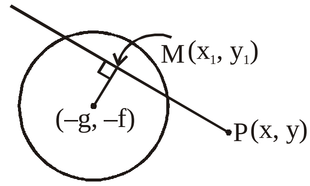
9. EQUATION OF THE CHORD WITH A GIVEN MID-POINT

The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid-point $M(x_1, y_1)$ is $y - y_1 = \frac{-x_1 + g}{y_1 + f}(x - x_1)$. This on simplification can be put in the form $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is obviously $T = S_1$.



10. IMPORTANT RESULTS TO REMEMBER

If $S = 0$ be a curve then $S_1 = 0$ indicates the equation which is obtained by substituting $x = x_1$ and $y_1 = y_1$ in the equation of the given curve, and $T = 0$ is the equation which is obtained by substituting x^2 by xx_1, y^2 by yy_1, $2xy$ by $xy_1 + yx_1$, $2x$ by $x + x_1$, $2y$ by $y + y_1$ in the equation $S = 0$.

If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ then $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + c$, and

$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

- Equation of the tangent to $x^2 + y^2 + 2gx + 2fy + c = 0$ at $A(x_1, y_1)$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.
- The condition that the straight line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$ and the point of contact is $(a^2m/c, a^2/c)$ i.e. $y = mx \pm a\sqrt{1+m^2}$ is always a tangent to the circle $x^2 + y^2 = a^2$ whatever be the value of m.
- The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + fy + c = 0$ is $T^2 = SS_1$.
- The equation of the chord of the circle $S = 0$, whose mid point (x_1, y_1) is $T = S_1$.
- The length of the tangent drawn from a point (x_1, y_1) outside the circle $S = 0$, is $\sqrt{S_1}$.
- The equation of the Chord of contact of a point (x_1, y_1) is $T = 0$.

Caution:

The equations of both the Chord of contact and the tangent are T = 0, which may cause confusion to the student. The student must realise that if the point lies on the circle, T = 0 gives the equation of tangent and if the point lies outside the circle, T = 0 gives the equation of Chord of contact.

11. DIRECTOR CIRCLE

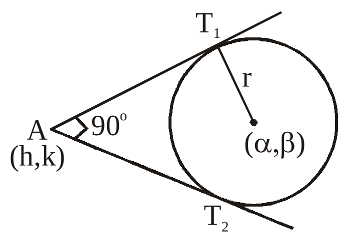
The locus of the point of intersection of perpendicular tangents is called director circle.

If $(x - \alpha)^2 + (y - \beta)^2 = r^2$ is the equation of a circle

AT_1 and AT_2 are two perpendicular tangents then locus of A is called director circle

equation of director circle is $(x - \alpha)^2 + (y - \beta)^2 = 2r^2$

So Director Circle of a circle is a concentric circle with radius equal to $\sqrt{2}$ times the radius of the original circle.



12. EQUATION OF A CHORD OF A CIRCLE

(A) Parametric Form

Consider the circle $x^2 + y^2 = a^2$ with its centre at the origin O and of radius a. Let P(x, y) be any point on the circle ; denote the vectorial angle of OP by f which can have any value between 0º and 360º. Then

$$x = a\cos\phi,\ y = a\sin\phi \qquad ....(1)$$

are the parametric equations of the circle, ϕ being the parameter. The equation of the circle in rectangular coordinates is at once obtained from (1) by elimination of ϕ, since

$$x^2 + y^2 = a^2(\cos^2\phi + \sin^2\phi) = a^2$$

The equation of the chord joining the two points whose parametric angles are ϕ_1 and ϕ_2 is

$$\frac{y - a\sin\phi_1}{a(\sin\phi_1 - \sin\phi_2)} = \frac{x - a\cos\phi_1}{a(\cos\phi_1 - \cos\phi_2)} \quad \text{or} \quad x(\sin\phi_1 - \sin\phi_2) - y(\cos\phi_1 - \cos\phi_2)$$

$$= a[\sin\phi_1\cos\phi_2 - \sin\phi_2\cos\phi_1] = a\sin(\phi_1 - \phi_2)$$

or $\quad 2x\sin\frac{1}{2}(\phi_1 - \phi_2)\cos\frac{1}{2}(\phi_1 + \phi_2) + 2y\sin\frac{1}{2}(\phi_1 - \phi_2)\sin\frac{1}{2}(\phi_1 + \phi_2)$

$$= 2a\sin\frac{1}{2}(\phi_1 - \phi_2)\cos\frac{1}{2}(\phi_1 - \phi_2)$$

or $\quad x\cos\frac{1}{2}(\phi_1 + \phi_2) + y\sin\frac{1}{2}(\phi_1 + \phi_2) = a\cos\frac{1}{2}(\phi_1 - \phi_2)$

(B) Another Form:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the extremities of a given chord of a circle. Then, the chord PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad ....(1)$$

We require to take account of the conditions that P and Q lie on the circle, considering first a circle of radius r with its centre at the origin and, second, a circle with its equation in general form.

(i) Circle, $x^2 + y^2 = r^2$.

Since P and Q lie on the circle, then

$$x_1^2 + y_1^2 = r^2 \text{ and } x_2^2 + y_2^2 = r^2 \qquad ....(2)$$

whence, by subtraction,

$$x_2^2 - x_1^2 = -(y_2^2 - y_1^2)$$

or $\quad \dfrac{y_2 - y_1}{x_2 - x_1} = -\dfrac{x_2 + x_1}{y_2 + y_1}$

Hence, (1) becomes

$$y - y_1 = -\frac{x_1 + x_2}{y_2 + y_1}(x - x_1)$$

or $\quad (x - x_1)(x_1 + x_2) + (y - y_1)(y_1 + y_2) = 0 \qquad ....(3)$

This is the equation of the chord.

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(ii) Circle, $x^2 + y^2 + 2gx + 2fy + c = 0$.

If P (x_1, y_1) and Q (x_2, y_2), now refer to the extremities of a chord of this circle, then

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \text{ and } x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$

whence, by subtraction,

$$x_2^2 - x_1^2 + 2g(x_2 - x_1) + y_2^2 - y_1^2 + 2f(y_2 - y_1) = 0$$

or $(x_2 - x_1)[x_2 + x_1 + 2g] + (y_2 - y_1)[y_2 + y_1 + 2f] = 0$

or $\dfrac{y_2 - y_1}{x_2 - x_1} = -\dfrac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f}$

Hence, by (1), the equation of the chord PQ is

$$(x - x_1)(x_1 + x_2 + 2g) + (y - y_1)(y_1 + y_2 + 2f) = 0 \qquad \text{........(4)}$$

Corollary:

The line from the centre of the circle to the mid-point of a chord is perpendicular to the chord.

We need consider (i) only. If M is the mid-point of the chord PQ, its coordinates are $[\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)]$ and the gradient of OM is $(y_1 + y_2)/(x_1 + x_2)$. The gradient of PQ is, by (3), $-(x_1 + x_2)/(y_1 + y_2)$. The product of the gradients is -1; hence OM is perpendicular to PQ the well-known result in pure geometry.

13. EQUATION OF THE LINES JOINING THE ORIGIN TO THE POINTS OF INTERSECTION OF A CIRCLE AND A LINE

(i) Circle, $x^2 + y^2 = r^2$.

We take the line to be

$$lx + my + n = 0 \text{ or } \frac{lx + my}{-n} = 1. \qquad \text{.......(1)}$$

We assume that (1) cuts the circle, the condition being that the radius, r, is greater than the perpendicular, p, from the origin to (1); thus since $p = \pm \dfrac{n}{\sqrt{l^2 + m^2}}$, then $r^2(l^2 + m^2) > n^2$.

Consider now the equation

$$x^2 + y^2 = r^2 \left(\frac{lx + my}{-n}\right)^2 \qquad \text{.......(2)}$$

or, in its expanded form,

$$(n^2 - l^2r^2)x^2 - 2lmr^2xy + (n^2 - m^2r^2)y^2 = 0 \qquad \text{.......(3)}$$

which is homogeneous equation of 2nd degree which represent a pair of straight line through origin.

(ii) Circle in general form:

Consider the equation

$$x^2 + y^2 + (2gx + 2fy)\left(\frac{lx + my}{-n}\right) + c\left(\frac{lx + my}{-n}\right)^2 = 0 \qquad \text{.......(5)}$$

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As in (i) this equation represents a line-pair joining the origin to the points of intersection of the circle and the line (1). In this (5) is the equation required.

Note : The procedure in both cases is to convert the equation of the circle into a homogeneous quadratic expression in x and y by means of the equation of the line written in the form $\frac{lx+my}{-n}$ = 1. Thus, in (i), we write the equation of the circle $x^2 + y^2 = r^2$ and replace the factor 1 by $\left(\frac{lx+my}{-1}\right)^2$.

Similarly, in (ii), we write the equation of the circle as

$$x^2 + y^2 + (2gx + 2fy).\ 1 + c.\ 1^2 = 0$$

and replace the first factor 1 by $\frac{lx+my}{-n}$ and the second factor 1 by $\left(\frac{lx+my}{-n}\right)^2$.

Illustration 23 : ***Find the equations of the lines joining the origin to the ends of the chord of intersection of the circle $x^2 + y^2 = 25$ and the line 7x + y = 25.***

Solution : In the prescribed form the line is written as $\frac{7x+y}{25}$ = 1. The required equation is

$$x^2 + y^2 = 25\left(\frac{7x+y}{25}\right)^2$$

which becomes

$$12x^2 + 7xy - 12y^2 = 0$$

or

$$(4x - 3y)(3x + 4y) = 0$$

The lines are:

$$4x - 3y = 0 \text{ and } 3x + 4y = 0 \qquad \text{......(1)}$$

It is easy to verify in this case that the ends of the chord of intersection are (3, 4) and (4, – 3) from which the lines (i) are at once derived.

EXERCISE - 10

Find the equations of the lines joining the origin to the ends of the chords of intersection of the following lines and circles -

(i) $2x + 3y = 6,\ x^2 + y^2 = 9$

(ii) $2x - y + 4 = 0,\ x^2 + y^2 + 2x - 2y = 8$

14. USE OF THE PARAMETRIC EQUATIONS (IN TRIGONOMETRICAL FORM) OF A STRAIGHT LINE INTERSECTING A CIRCLE

The parametric equations of a straight line provide a powerful method for dealing with an important class of problems in analytical geometry relating generally to the circle and conics ; the method is most simply illustrated with reference to the circle.

(i) AFG is a line, of angle of slope θ, passing through a given point A (h, k) and intersecting the given circle $x^2 + y^2 = r^2$ at F and G, A being external to the circle. If (x, y) are the coordinates of any point P on the line, then

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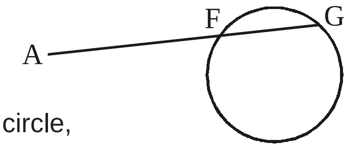
$x = h + R\cos\theta,\ y = k + R\sin\theta$ .....(1)



where R denotes the vector $\overrightarrow{AP}$. If the point P is on the circle, the coordinates given by (1) satisfy the equation of the circle so that

$(h + R\cos\theta)^2 + (k + R\sin\theta)^2 = r^2$

or $R^2 + 2R(h\cos\theta + k\sin\theta) + h^2 + k^2 - r^2 = 0$ ....(2)

This is a quadratic equation in R giving in general two values, R_1 and R_2, for R relating to the two points F and G at which the line cuts the circle.

The product of the roots R_1 and R_2 of (2) is given by

$R_1R_2 = h^2 + k^2 - r^2$,

that is, $\overrightarrow{AF}\cdot\overrightarrow{AG} = h^2 + k^2 - r^2$ ....(3)

Also, for later use,

$R_1 + R_2 = -2(h\cos\theta + k\sin\theta)$ ....(4)

Since the right-hand side of (3) is independent of θ, there follows the geometrical theorem that, for all secants to the circle from an external point A, AF. AG is constant. Further, if AT is the tangent from A to the circle then, since $h^2 + k^2 = OA^2$, we have $h^2 + k^2 - r^2 = AT^2$. The theorem may then be stated in the well-known form

$\overrightarrow{AF}\cdot\overrightarrow{AG} = AT^2$.

(ii) When the given point, say B, is within the circle (fig.) then from (2),

$\overrightarrow{BF}\cdot\overrightarrow{BG} = h^2 + k^2 - r^2 = OB^2 - r^2$ ....(5)

where h and k now denote the coordinates of B and OB < r.

Since the right-hand side of (4) is negative, $\overrightarrow{BF}$ and $\overrightarrow{BG}$ are of opposite sign. In terms of the lengths of the segments BF and BG, (5) becomes

$BF.\ BG = r^2 - OB^2$

This is the analytical expression of the theorem in pure geometry that the product of the lengths of segments of chord through a given point B is the same for all chords through B.

(iii) If the given point is taken to be the mid-point, M, of the Chord FG, then $\overrightarrow{MG} = -\overrightarrow{MF}$ or $R_1 = -R_2$ or $R_1 + R_2 = 0$. If (h, k) now denote the coordinates of M we have, from (4),

$R_1 + R_2 = -2(h\cos\theta + k\sin\theta) = 0$

Hence the locus of the mid-points of parallel chords is

$x\cos\theta + y\sin\theta = 0$, ...(6)

which is a straight line through the origin and perpendicular to the parallel chords; this, again, is the analytical expression of the well-known theorem in pure geometry.

Illustration 24 : ***The secant AFG from A $(-7\frac{1}{4}, -7)$ of gradient 3/4, cuts the circle $x^2 + y^2 = 25$ at F and G. Find the coordinates of the mid-point M of the chord FG.***

Solution : A sketch shows that the line through A with the given gradient will actually intersect the circle and that if 2c denotes the length of the chord FG.

$|\overrightarrow{AF}| = |\overrightarrow{AM}| - c,\ |\overrightarrow{AG}| = |\overrightarrow{AM}| + c$ ....(1)

The equation for R is

$$R^2 + 2R(-7\frac{1}{4}\cos\theta - 7\sin\theta) + (7\frac{1}{4})^2 + 49 - 25 = 0$$

Since $\cos\theta = 4/5$, $\sin\theta = 3/5$,

$$\therefore \quad R^2 - 20R + \frac{1225}{16} = 0 \qquad ....(2)$$

The roots are real, since $(10)^2 > \frac{1225}{16}$; accordingly, the line cuts the circle in two points, which substantiates analytically the inference from the sketch.

From (ii) $R_1 + R_2 = 20$ or $|\overrightarrow{AF}| + |\overrightarrow{AG}| = 20$.

Hence, from (i), $|\overrightarrow{AM}| = 10$. The coordinates of M are then :

$(-7\frac{1}{4} + 10\cos\theta, -7 + 10\sin\theta)$ or $(\frac{3}{4}, -1)$.

15. RADICAL AXIS

The radical axis of two circles is the locus of a point from which the tangent segments to the two circles are of equal length.

Equation of the Radical Axis:

In general S – S' = 0 represents the equaion of the radical Axis to the two circles i.e. $2x(g - g') + 2y(f - f') + c - c' = 0$, where $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$.

- If S = 0 and S' = 0 intersect in real and distinct point then S – S' = 0 is the equation of the common chord of the two circles.
- If S = 0 and S′ = 0 touch each other, then S – S' = 0 is the equation of the common tangent to the two circles at the point of contact.

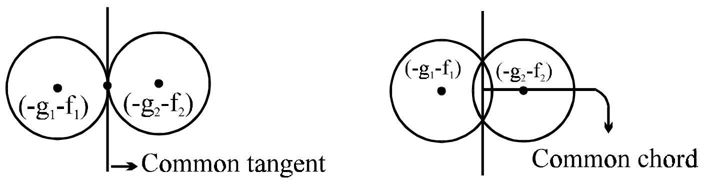


Illustration 25 : ***Prove that the circle $x^2 + y^2 - 6x - 4y + 9 = 0$ bisects the circumference of the circle $x^2 + y^2 - 8x - 6y + 23 = 0$.***

Solution : The given circles are

$$S_1 \equiv x^2 + y^2 - 6x - 4y + 9 = 0 \qquad ....(1)$$

and $$S_2 \equiv x^2 + y^2 - 8x - 6y + 23 = 0 \qquad ....(2)$$

Equation of the common chord of circles (1) and (2) which is also the radical axis of the circles S_1 and S_2 is

$$S_1 - S_2 = 0 \text{ or, } 2x + 2y - 14 = 0$$

or, $$x + y - 7 = 0 \qquad ....(3)$$

Centre of the circle S_2 is (4, 3). Clearly, line (3) passes through the point (4, 3) and hence line (3) is the equation of the diameter of the circle (2). Hence circle (1) bisects the circumference of circle (2).

15.1 THE RADICAL AXIS OF A PAIR OF CIRCLES

If Q is the centre of a circle S of radius r and P (x, y) is any point, the power of P which we denote by W with respect to the circle S is defined by

$$W = PQ^2 - r^2 \qquad ......(1)$$

If P is outside the circle then PQ > r and W is positive; if P is within the circle, PQ < r and W is negative.

The radical axis of two circles S_1 (centre Q_1, radius r_1) and S_2 (centre Q_2, radius r_2) is defined to be the locus of P such that the power of P with respect to the circle S_1 is equal to the power of P with respect to the circle S_2 or, from (1),

$$PQ_1^2 - r_1^2 = PQ_2^2 - r_2^2 \qquad ......(2)$$

We consider, first, the case of two non-intersecting circles and, second, the case of two intersecting circles.

(i) Non-intersecting circles:

In fig. if P(x, y) is a point on the radical axis of the circles S_1 and S_2 then, from (2),

$$(x + g)^2 + (y + f)^2 - (g^2 + f^2 - c) = (x + G)^2 + (y + F)^2 - (G^2 + F^2 - C)$$

or $\quad 2(g - G)x + 2(f - F)y + c - C = 0 \qquad ....(3)$

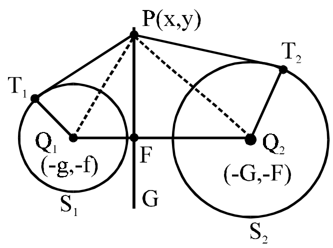
This is the equation of the radical axis, represented by the line PFG.

But (3) is equivalent to

$$S_1 - S_2 = 0$$

where $S_1 = 0$ and $S_2 = 0$ are the equations of the two circles. Thus, the radical axis of two non-intersecting circles is obtained by subtracting the equations of the circles, these being supposed to be expressed in the prescribed form.

If PT_1 and PT_2 are the tangents from P to S_1 and S_2, then $W_1 \equiv PQ_1^2 - r_1^2 = PT_1^2$; similarly, $W_2 = PT_2^2$.



Thus, in the case of non-intersecting circles the radical axis is the locus of a point P such that the lengths of the tangents from P to S_1 and S_2 are equal.

In fig. F is the point of intersection of the radical axis and the line Q_1Q_2; denote Q_1F by u and Q_1Q_2 by d. Since F is on the radical axis.

$$FQ_1^2 - r_1^2 = FQ_2^2 - r_2^2 \qquad \text{or} \qquad u^2 - r_1^2 = (d - u)^2 - r_2^2$$

from which $\quad 2du = d^2 + r_1^2 - r_2^2$

Similarly, $\quad 2dv = d^2 - r_1^2 + r_2^2$,

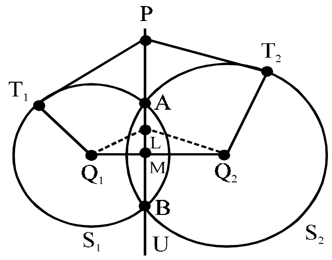
where $\quad v = Q_2F$

Thus, F divides Q_1Q_2 in the ratio

$$(d^2 + r_1^2 - r_2^2) : (d^2 - r_1^2 + r_2^2).$$

The radical axis is external to each of the non-intersecting circles.

(ii) Intersecting circles:



In fig. AB is the common chord. Let P be any point on BA produced and PT_1, PT_2 the tangents from P to the circles. Then

$$PT_1^2 = PA.\ PB = PT_2^2\ ;$$

hence, $\quad PQ_1^2 - r_1^2 = PQ_2^2 - r_2^2$

or, by (1), $W_1 = W_2$. Accordingly, P is a point on the radical axis.

Consider now a point L between A and B; let

Q_1Q_2 cut AB at M. Now, LM is perpendicular to Q_1Q_2; hence

$$LM^2 = Q_1L^2 - Q_1M^2 = Q_1L^2 - (r_1^2 - AM^2).$$

Similarly, $LM^2 = Q_2L^2 - (r_2^2 - AM^2)$; hence

$$Q_1L^2 - r_1^2 = Q_2L^2 - r_2^2$$

and accordingly the powers of L with respect to the circles are equal so the L is a point on the radical axis.

For A, the power W_1 is $Q_1A^2 - r_1^2$, that is, zero; similarly, for B. Thus for two intersecting circles the radical axis is the line PU of the common chord whose equation is,

$$2(g - G)x + 2\ (f - F)y + c - C = 0.$$

(iii) Summary:

For all pairs of circles the radical axis is the line

$$2(g - G)x + 2\ (f - F)y + c - C = 0$$

obtained by subtracting the equations of the two circles, given in the standard forms.

The radical axis is perpendicular to the line joining the centres of the circles.

If the circles are intersecting, the radical axis is the line passing through the points of intersection and is thus the common chord extended both ways.

15.2 THE RADICAL AXES OF THREE CIRCLES, TAKEN IN PAIRS, ARE CONCURRENT

Let the equations of the three circles S_1, S_2 and S_3 be

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

and $\quad S_3 \equiv x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0 \quad$ ....(1)

Now, by the previous section, the radical axis of S_1 and S_2 is obtained by subtracting the equations of these circles ; hence it is

$$S_1 - S_2 = 0 \quad ....(2)$$

Similarly, the radical axis of S_2 and S_3 is

$$S_2 - S_3 = 0 \quad ....(3)$$

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The lines (2) and (3) meet at a point whose coordinates say, (X, Y) satisfy

$S_1 – S_2 = 0$ and $S_2 – S_3 = 0$;

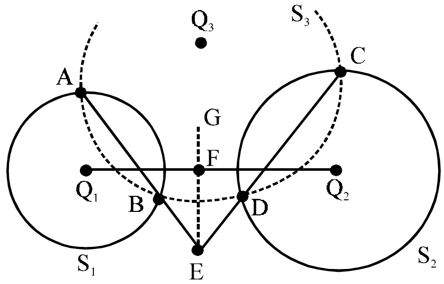
hence the coordinates (X, Y) satisfy

$(S_1 – S_2) + (S_2 – S_3) = 0$;

that is , (X, Y) satisfy

$S_1 – S_3 = 0$ ....(4)

But (4) is the radical axis of the circles S_1 and S_3 and hence the three radical axes are concurrent. The point of concurrency of the three radical axes is called the radical centre.



15.3 CONSTRUCTION OF THE RADICAL AXIS OF TWO GIVEN CIRCLES

(i) Intersecting circle:
The radical axis is the line defined by the common chord and no further comment is necessary.

(ii) Non-intersecting circles:
In figure, the given circles are S_1 and S_2; their centres are Q_1 and Q_2. We make use of the theorem of the previous section.

Draw any circle, S_3, to cut both S_1 and S_2; its centre is Q_3. Since S_1 and S_3 intersect at A and B the radical axis is the common chord produced both ways, that is, the line ABE; similarly, the radical axis of S_2 and S_3 is CDE. The point of intersection of these radical axes is E; hence, by the previous section, E is a point on the radical axis of S_1 and S_2. Further, this radical axis is perpendicular to the line of centres of S_1 and S_2, that is, the line Q_1Q_2. The radical axis of S_1 and S_2 is then the line EFG which is drawn perpendicular to Q_1Q_2 through E.

16. FAMILY OF CIRCLES

- If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$ are two intersecting circles, then $S + \lambda S' = 0$, $\lambda \neq – 1$, is the equation of the family of circles passing through the points of intersection $S = 0$ and $S' = 0$.
- If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle which is intersected by the straight line $\mu \equiv ax + by + c = 0$ at two real and distinct points, then $S + \lambda\mu = 0$ is the equation of the family of circles passing through the points of intersection of $S = 0$ and $\mu = 0$. If $\mu = 0$ touches $S = 0$ at P, then $S + \lambda\mu = 0$ is the equation of the family of circles, each touching $\mu = 0$ at P.
- The equation of a family of circles passing through two given points (x_1, y_1) and (x_2, y_2) can be written in the form.

$$(x–x_1)(x– x_2) + (y – y_1)(y – y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0, \text{ where } \lambda \text{ is a parameter.}$$

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- The equation of the family of circles which touch the line $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any value of m is $(x - x_1)^2 + (y - y_1)^2 + \lambda [(y - y_1) - m (x - x_1)] = 0$. If m is infinite, the equation is $(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$.
- The two circles are said to intersect orthogonally if the angle of intersection of the circles i.e., the angle between their tangents at the point of intersection is 90º. The condition for the two circles $S = 0$ and $S_1 = 0$ to cut each other orthogonally is $2gg_1 + 2ff_1 = C + C_1$.

Illustration 26 : ***Find the equation of the circle through the points A (12, 4), B(8, 12) and C(–6, – 2).***

Solution : The equation of AB is easily seen to be $2x + y - 28 = 0$.

The equation of any circle through A and B is,

$$(x - 12)(x - 8) + (y - 4)(y - 12) + \lambda (2x + y - 28) = 0 \qquad ....(1)$$

If C lies on this circle, then

$$(-6 - 12)(-6 - 8) + (-2 - 4)(-2 - 12) + \lambda(-12 - 2 - 28) = 0$$

whence $\lambda = 8$. Insert this value of λ in (1); then, the equation of the circle through A, B and C is

$$x^2 - 20x + 96 + y^2 - 16y + 48 + 16x + 8y - 224 = 0$$

$$x^2 + y^2 - 4x - 8y - 80 = 0,$$

Illustration 27 : ***Obtain the equation of the circumcircle of the triangle whose sides are 5x + y – 103 = 0, x – y – 11 = 0 and 2x – 3y – 31 = 0 and find its radius.***

Solution : The curve

$$(5x + y - 103)(x - y - 11) + \lambda(x - y - 11)(2x - 3y - 31)$$
$$+ \mu (2x - 3y - 31)(5x + y - 103) = 0 \qquad ....(1)$$

λ, μ being constants, passes through the verticies of the triangle, for the coordinates of the point of intersection of any two of the given lines which make two of the expressions $5x + y - 103$, $x - y - 11$, $2x - 3y - 31$ simultaneously zero, are easily seen to satisfy the equation to the curve.

Here λ and μ are so chosen that the curve represents a circle.

For that, coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$.

simplifying equation (1), we get

$$(5 + 2\lambda + 10\mu) x^2 + (-4 - 5\lambda - 13\mu) xy + (-1 + 3\lambda - 3\mu) y^2$$
$$+ (-158 - 53\lambda - 361\mu) x + (92 + 64\lambda + 278\mu) y$$
$$+ (1133 + 341\lambda + 3193\mu) = 0.$$

$$5 + 2\lambda + 10\mu = -1 + 3\lambda - 3\mu \text{ and } -4 - 5\lambda - 13\mu = 0$$

i.e., $\lambda - 13\mu = 6$ and $5\lambda + 13\mu = -4$.

Solving these two equations, we get $\lambda = \frac{1}{3}$ and $\mu = -\frac{17}{39}$

Substituting the values of λ and μ in the equation of the circle, we get

$$\frac{51}{39}x^2 + \frac{51}{39}y^2 - \frac{714}{39}x - \frac{306}{39}y - \frac{5661}{39} = 0$$

i.e., $x^2 + y^2 - 14x - 16y - 111 = 0$.

Alternate method. We can find the coordinates of the vertices of the triangle and from that the equation of the circle passing through those points.

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This analytical definition differs slightly from the elementary school definition where a circle refers to the region bounded by a curve of the above form and according to the earlier notion, the circumference is what is meant by the circle according to the analytical definition. To distinguish the two in higher mathematics, a plane area bounded by a circle is called a disc.

REMARK

Taking 'a' as a parameter, the equation $(x - x_1)^2 + (y - y_1)^2 = a^2$ represents a family of concentric circles.

Do your self:

Find the equation of the circle described on the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $2x^2 + 2y^2 + 8x + 6y + 4 = 0$ as diameter.

Ans. $2(x^2 + y^2) + 2x + 6y + 1 = 0$.

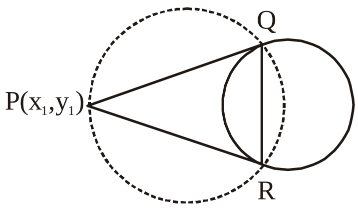
Illustration 28 : ***Tangents PQ and PR are drawn to the circle $x^2 + y^2 = a^2$ from the point $P(x_1, y_1)$. Prove that the equation of the circumcircle of ΔPQR is $x^2 + y^2 - xx_1 - yy_1 = 0$.***

Solution : QR is the chord of contact of the tangents to the circle

$$x^2 + y^2 - a^2 = 0 \qquad ....(1)$$

equation of

$$QR \text{ is } xx_1 + yy_1 - a^2 = 0 \qquad ....(2)$$



The circumcircle of ΔPQR is a circle passing through the intersection of the circle (1) and the line (2) and the point $P(x_1, y_1)$.

Circle through the intersection of (1) and (2) is

$$x^2 + y^2 - a^2 + \lambda(xx_1 + yy_1 - a^2) = 0 \qquad ....(3)$$

it will pass through (x_1, y_1) if

$$x_1^2 + y_1^2 - a^2 + \lambda(x_1^2 + y_1^2 - a^2) = 0$$

$$\lambda = -1 \text{ (since } x_1^2 + y_1^2 \neq a^2)$$

Hence equation of circle is $(x^2 + y^2 - a^2) - (xx_1 + yy_1 - a^2) = 0$

or $\quad x^2 + y^2 - xx_1 - yy_1 = 0$.

EXERCISE - 11

1. Find the equation of the circle which passes through (3, 1) and the points of intersection of the circles $(x - 2)^2 + (y + 3)^2 = 5$, $(x + 4)^2 + (y - 2)^2 = 34$.
2. Find the equation of the circle with the common chord of the circles $x^2 + y^2 - 2x - 7y + 7 = 0$, $x^2 + y^2 - 6x - 5y + 9 = 0$ as diameter.

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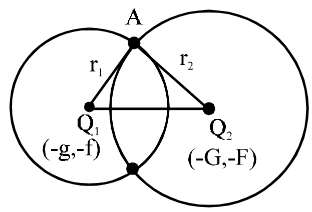
17. ORTHOGONAL CIRCLES

Two circles are said to be orthogonal if the tangents to the circles at either point of intersection are at right angles.



In fig. Q_1 and Q_2 are the centres of the circles

$$S_1 \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ....(1)$$

and $S_2 \equiv x^2 + y^2 + 2Gx + 2Fy + C = 0$ ....(2)

the circles, S_1 and S_2, intersect at A and B.

The tangent at A to the circles S_1 is perpendicular to the radius Q_1A, and the tangent at A to S_2 is perpendicular to the radius Q_2A. Hence, if the two tangents are at right angles, then the radii Q_1A and Q_2A must also be at right angles. Accordingly, the condition that S_1 and S_2 should be orthogonal is that $\angle Q_1AQ_2$ should be 90º; by Pythagoras' theorem this condition is equivalent to

$$Q_1Q_2^2 = Q_1A^2 + Q_2A^2 = r_1^2 + r_2^2 \qquad ....(3)$$

or $(g - G)^2 + (f - F)^2 = g^2 + f^2 - c + G^2 + F^2 - C$

or, on simplification,

$$2(gG + fF) = c + C. \qquad ....(4)$$

Since, AQ_2 is perpendicular to the radius Q_1A, the tangent at A to the circle S_1 passes through the centre of the circle S_2; similarly, the tangent at A to S_2 passes through the centre of S_1. In numerical examples the procedure of solution should be based on the condition expressed by (3).

Illustration 29 : ***Show that the circle passing through the origin and cutting the circles $x^2 + y^2 - 2a_1x - 2b_1y + c_1 = 0$ and $x^2 + y^2 - 2a_2x - 2b_2y + c_2 = 0$ orthogonally is***

$$\begin{vmatrix} x^2+y^2 & x & y \\ c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \end{vmatrix} = 0.$$

Solution : Let the equation of the circle passing through the origin be

$$x^2 + y^2 + 2gx + 2fy = 0 \qquad ....(1)$$

It cuts the given two circles orthogonally $\Rightarrow -2ga_1 - 2fb_1 = c_1$

$\Rightarrow \quad c_1 + 2ga_1 + 2fb_1 = 0$ ....(2)

and $\quad -2ga_2 - 2fb_2 = c_2$

$\Rightarrow \quad c_2 + 2ga_2 + 2fb_2 = 0$ ....(3)

Eliminating 2f and 2g from (1), (2) and (3), we get

$$\begin{vmatrix} x^2+y^2 & x & y \\ c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \end{vmatrix} = 0.$$

Illustration 30 : ***Prove that the circles $x^2 + y^2 - 6x + 8y + 10 = 0$ and $x^2 + y^2 + 8x + 10y + 6 = 0$ are orthogonal.***

Solution : The centre, Q_1, of the first circle, S_1 is (3, – 4) and the radius, r_1, is given by $r_1^2 = 15$.

The centre, Q_2, of the second circle, S_2, is (–4, – 5) and the radius is given by $r_2^2 = 35$.

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Now, $Q_1Q_2^2 = 7^2 + 1^2 = 50$ and $r_1^2 + r_2^2 = 50$; the condition (3) is satisfied and, consequently, the circles are orthogonal.

It is easily verified that the condition given by (4) is satisfied, for we have

$$g = -3,\ G = 4;\ f = 4,\ F = 5;\ c = 10,\ C = 6,$$

and $\quad 2(gG + fF) = 2(-12 + 20) = 16 \equiv c + C$.

EXERCISE - 12

1. Prove that the circles $x^2 + y^2 + 6x + 2y = 15$ and $x^2 + y^2 - 14x + 12y = 15$ cut orthogonally.
2. Find the equation of the circle which cuts the circles $x^2 + y^2 = 6y$, $x^2 + y^2 - 4x - 4y = 1$ orthogonally and which passes through $(-1, 2)$.

18. THE COMMON CHORD OF TWO INTERSECTING CIRCLES

We take the equations of the two circles in fig. to be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ....(1)$$

and $x^2 + y^2 + 2Gx + 2Fy + C = 0$ ....(2)

Let S_1 denote the left-hand side of (1) and S_2 the left-hand side of (2) ; then the equations of the two circles are written in the compact form

$$S_1 = 0 \text{ and } S_2 = 0. \qquad ....(3)$$

It is convenient to refer to the circle whose equation is $S_1 = 0$ as the circle S_1; similarly, for the second circle.

Let AB in Fig. the common chord and let (X, Y) be the coordinates of A. Then, since A lies on S_1 and S_2, its coordinates satisfy (1) and (2) so that

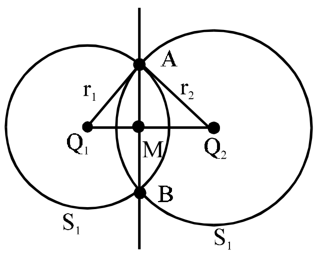
$$X^2 + Y^2 + 2gX + 2fY + c = 0$$

and $X^2 + Y^2 + 2GX + 2FY + C = 0$

whence, by subtraction,

$$2(g - G)X + 2(f - F)Y + c - C = 0.$$

But this is the condition that A lies on the line



$$2(g - G)\,x + 2\,(f - F)\,y + c - C = 0. \qquad ....(4)$$

Similarly, B lies on the line (4). Accordingly, (4) is the equation of the common chord, AB, of the two intersecting circles, ;it is obtained by subtracting the equations of the two circle; it is obtained by subtracting the equations of the two circles when expressed in the forms (1) and (2) that is, when the coefficients of x^2 and y^2 are unity in both equations.*

The equation (4) the equation of the common chord of two intersecting circle is, in compact form,

$$S_1 - S_2 = 0. \qquad ....(5)$$

If one equation, for example, is $A(x^2 + y^2) + 2G'x + 2F'y + C' = 0$ we divide throughout by A so as to the standard form.

Corollary:

The common chord is perpendicular to the line joining the centres of the circles.

The gradient of the common chord is, from (4), equal to $-(g - G)/(f - F)$ and that of Q_1Q_2 is $(f - F)/(g - G)$; the product of the gradients is -1 and the well-known theorem in pure geometry is established.

Note :

It is to be remarked that (4) is obtained by subtracting the equations of the two circles (in the prescribed form) and that this operation can be performed irrespective of whether the circles do, or do not, intersect. The interpretation of (4) as the common chord applies only when the circles intersect; we shall deal later with the geometrical significance of (4) when the circles do not intersect.

Illustration 31 : ***If two circles cut a third circle orthogonally, prove that their common chord will pass through the centre of the third circle.***

Solution : Let us take the equation of the two circles as

$$x^2 + y^2 + 2\lambda_1 x + a = 0 \qquad ....(1)$$

$$x^2 + y^2 + 2\lambda_2 x + a = 0 \qquad ....(2)$$

We can select axes suitable (the line of centers as x-axis and the point midway between the centre as origin) to get the above form of equation.

Let the third circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ....(3)$$

The circle (1) and (3) cut orthogonally

$$2\lambda_1 g = a + c \qquad ....(4)$$

The circles (2) and (3) cut orthogonally

$$2\lambda_2 g = a + c \qquad ....(5)$$

From (4) and (5), $2g(\lambda_1 - \lambda_2) = 0$ but $\lambda_1 \neq \lambda_2$

$\therefore \quad g = 0$

Hence centre of the third circle $(0, -f)$

The common chord of (1) and (2) has the equation $S_1 - S_2 = 0$

i.e. $\quad x^2 + y^2 + 2\lambda_1 x + a - (x^2 + y^2 + 2\lambda_2 x + a) = 0$

or $\quad 2(\lambda_1 - \lambda_2)x = 0 \therefore x = 0$

$\therefore \quad (0, -f)$ satisfies the equation $x = 0$.

DO YOURSELF:

What about the radical axis of two concentric circles or one circle lying completely inside the other circle without touching ? **Ans.** Radical axis will be at infinity.

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EXERCISE - 13

1. Find the tangent of the acute angle between the tangents at either point of intersection of the circles

   $x^2 + y^2 - 5x + 4y + 8 = 0$, $x^2 + y^2 + 5x - 6y = 22$

2. Prove that the length of the common chord of the circles $x^2 + y^2 - 6x = 0$, $x^2 + y^2 - 8y = 0$ is 24/5.

3. Prove that the circles $x^2 + y^2 = 10$, $x^2 + y^2 + 2x = 12$, $x^2 + y^2 - 2x = 8$ have a common chord; give its equation and the points of intersection with the circles.

4. Prove that the circles $x^2 + y^2 + 4x + 6y = 18$, $x^2 + y^2 - 2x - 3y + 3 = 0$, $x^2 + y^2 - 6x - 9y + 17 = 0$ have two points in common and find the equation of the common chord.

19. POSITION OF A CIRCLE WITH RESPECT TO A CIRCLE AND COMMON TANGENTS

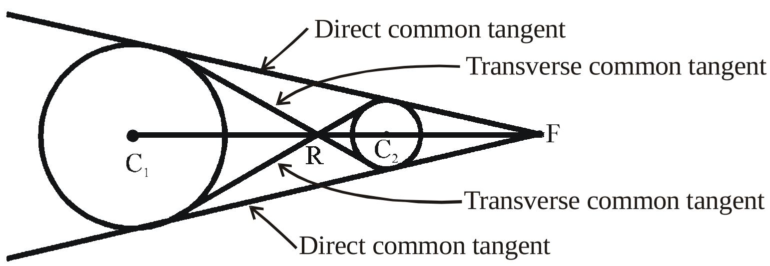
Before discussing position of two circles, let us first discuss common tangents.

1. **Direct Common tangent:**

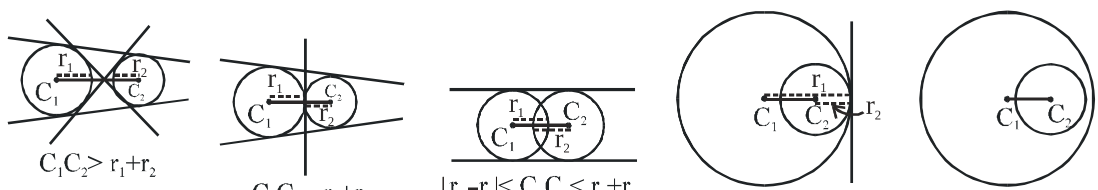
   Direct common tangent is a tangent touching two circles at different points and not intersecting the line of centres between the centres as shown in figure. The direct common tangents to two circles meet on the line of centres and divide it externally in the ratio of the radii.

2. **Transverse common tangents:**

   Transverse common tangent is a tangent touching two circles at different points and intersecting the line of centres between the centres as shown in figure. The transverse common tangents also meet on the line of centres and divide it internally in the ratio of the radii.



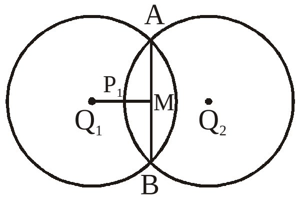
Regarding the position of a circle with respect to a circle, five situations are there



$C_1C_2 > r_1+r_2$	$C_1C_2 = r_1+r_2$	$\lvert r_1 - r_2\rvert < C_1C_2 < r_1+r_2$	$C_1C_2 = \lvert r_1 - r_2\rvert$	$0 \leqslant C_1C_2 < \lvert r_1 - r_2\rvert$
Circles do not intersect	Circles touch each other externally	Circles intersect in two points	Circles touch each other internally	One Circle lies completely inside other
Four common tangents can be drawn - two direct & two transverse	Three common tangents can be drawn	Two common tangents can be drawn	Only one common tangent can be drawn	No common tangent can be drawn

19.1 THE CONDITION THAT TWO CIRCLES SHOULD INTERSECT

In fig. the centre of the circles S_1 and S_2 are $Q_1(-g, -f)$ and $Q_2(-G, -F)$; their radii are r_1 and r_2. Let $p_1 \equiv Q_1M$ be the perpendicular to AB from Q_1. Then, for intersecting circles we must have



$p_1 < r_1$ ....(1)

or, alternatively, $p_2 < r_2$, where $p_2 \equiv Q_2M$ be the perpendicular from $(-G, -F)$ to the line

$S_1 - S_2 \equiv 2(g - G)x + 2(f - F)y + c - C = 0,$

obtained by subtracting the equations of the two circles in standard form.

Note :

A necessary condition for intersection is $r_1 + r_2 > Q_1Q_2$,. But this is not a sufficient condition; for example, the circles $(x - 1)^2 + (y - 1)^2 = 1$ and $x^2 + y^2 = 25$ are such that

$r_1 + r_2(\equiv 1 + 5) > Q_1Q_2(\equiv \sqrt{2})$;

but these circles do not intersect, for the first circle lies wholly within the second circle.

Illustration 32 : ***Prove that the circles***

$$x^2 + y^2 - 6x - 6y - 7 = 0 \text{ and } x^2 + y^2 - 10x + 7 = 0$$

intersect and find the coordinates of the common points.

Solution : The first circle, S_1, is $(x - 3)^2 + (y - 3)^2 = 25$; its centre Q_1 is (3, 3) and its radius is 5. The second circle, S_2, is $(x - 5)^2 + y^2 = 18$; its centre Q_2, is (5, 0) and its radius is $3\sqrt{2}$.

Subtracting the equations of the circles we obtain, after division by 2,

$2x - 3y - 7 = 0$ ....(1)

Then p_1, the perpendicular distance from $Q_1(3, 3)$ to the line (i), is $10/\sqrt{13}$ which is less than r_1 ($\equiv 5$).

Similarly p_2, the perpendicular distance from $Q_2(5, 0)$ to (i), is $3/\sqrt{13}$ which is less than r_2 ($\equiv 3\sqrt{2}$). Accordingly, the circles intersect and (i) is the equation of the common chord AB.

The coordinates of A and B are obtained by solving (i) and one of the equations of the circles in this case the second equation is the simpler for this purpose. On eliminating y between the two equations concerned we obtain

$9x^2 + (2x - 7)^2 - 90x + 63 = 0$

or $13x^2 - 118x + 112 = 0$ or $(x - 8)(13x - 14) = 0$.

Thus the abscissae of A and B are 8 and 14/13; from (i) the corresponding ordinates are 3 and –21/13; A and B are the points (8, 3) and (14/13, – 21/13).

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Illustration 33 : ***Do the circles***

$$x^2 + y^2 - 10x - 11 = 0,\ x^2 + y^2 + 14x - 10y + 73 = 0 \text{ intersect ?}$$

Solution : The first circle, S_1, is $(x - 5)^2 + y^2 = 36$; its centre, Q_1 is (5, 0) and its radius is 6.
By subtraction given one equation from other we obtain

$$12x - 5y + 42 = 0 \qquad \text{........(1)}$$

The perpendicular distance of $Q_1(5, 0)$ to (1) is given by

$$p_1 = \frac{102}{13} = 7\frac{11}{13},$$

which is greater than r_1 ($\equiv 6$). The criterion, $p_1 < r_1$, for intersection does not hold and we conclude that the two circles do not intersect.

It is readily verified that the solution of (1) and either given circles equation does not lead to real values of x and y .

EXERCISE - 14

1. Prove that the circles $x^2 + y^2 - 4x + 6y = 4$ and $x^2 + y^2 + 6x - 4y = 4$ intersect and that the length of the common chord is $3\sqrt{2}$ and also find the equations of the common chord and the coordinates of the common points.

Illustration 34 : ***Prove that the circles S_1 and S_2 given by***

$$x^2 + y^2 - 2x - 8y + 7 = 0 \text{ and } x^2 + y^2 + x - 9y - 2 = 0 \qquad \text{........(1)}$$

touch internally, and find the coordinates of the point of contact.

Solution : The centre of S_1 is (1, 4) and its radius is $\sqrt{10}$.

The centre of S_2 is $(-\frac{1}{2}, 4\frac{1}{2})$ and its radius is $\frac{3}{2}\sqrt{10}$.

The distance between the centres is $\sqrt{\left[\left(1\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right]}$ or $\frac{1}{2}\sqrt{10}$, and this is equal to the difference of the radii. Hence the circles touch internally. The common tangent is found by subtracting the equations (i); its equation is thus

$$3x - y - 9 = 0 \qquad \text{.......(2)}$$

The gradient of the line joining the centres is $\left(4\frac{1}{2} - 4\right)/\left(-\frac{1}{2} - 1\right)$ or $-1/3$ and the equation of the line is $y - 4 = -\frac{1}{3}(x - 1)$ or $x + 3y - 13 = 0$ ....(3)

The point of contact, A, is the point common to (2) and (3); by solution of these equations A is found to be (4, 3).

Illustration 35 : **Examine whether the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally.**

Solution : Let C_1 and C_2 be the centre of the circles.

$\Rightarrow$ $C_1 \equiv (1, 2)$ and $C_2 \equiv (0, 4)$. Let r_1 and r_2 be the radii of the circles

$\Rightarrow$ $r_1 = \sqrt{5}$ and $r_2 = 2\sqrt{5}$

Also $C_1C_2 = \sqrt{1+4} = \sqrt{5} = \gamma_1 - \gamma_2$.

Hence the circles touch each other internally.

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Illustration 36 : ***Prove that $x^2 + y^2 = a^2$ and $(x - 2a)^2 + y^2 = a^2$ are two equal circles touching each other. Find the equation of circle (or circles) of the same radius touching both the circles.***

Solution : Given circles are

$x^2 + y^2 = a^2$ ....(1)

and $(x - 2a)^2 + y^2 = a^2$ ....(2)

Let A and B be the centres and r_1 and r_2 the radii of the circles (1) and (2) respectively. Then

$A \equiv (0, 0)$, $B \equiv (2a, 0)$, $r_1 = a$, $r_2 = a$

Now $AB = \sqrt{(0-2a)^2 + 0^2} = 2a = r_1 + r_2$

Hence the two circles touch each other extenally.

Let the equation of the circle having same radius 'a' and touching the circles (1) and (2) be $(x - \alpha)^2 + (y - \beta)^2 = a^2$ ....(3)

Its centre C is (α, β) and radius $r_3 = a$

Since circle (3) touches the circle (1),

$AC = r_1 + r_3 = 2a$

$\Rightarrow$ $AC^2 = 4a^2 \Rightarrow \alpha^2 + \beta^2 = 4a^2$ ....(4)

Again since circle (3) touches the circle (2)

$BC = r_2 + r_3 \Rightarrow BC^2 = (r_2)$

$\Rightarrow$ $(2a - \alpha)^2 + \beta^2 = (a + a)^2$ $\Rightarrow \alpha^2 + \beta^2 - 4a\,\alpha = 0$

$\Rightarrow$ $4a^2 - 4a\,\alpha = 0$ [from (4)]

$\Rightarrow$ $\alpha = a$ and from (4), we have $\beta = \pm\sqrt{3}\,a$.

Hence, the required circles are

$(x - a)^2 + (y \mp a\sqrt{3})^2 = a^2$

or $x^2 + y^2 - 2ax \mp 2\sqrt{3}\,ay + 3a^2 = 0$.

EXERCISE - 15

1. Prove that the following pairs of circles touch, and find the equation of the common tangent and the coordinates of the point of contact -

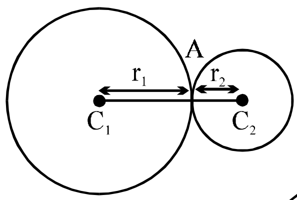
   (i) $x^2 + y^2 = 2$, $x^2 + y^2 + 3(x - y) = 8$ (ii) $x^2 + y^2 + 4x - 6y = 3$, $x^2 + y^2 - 12x - 18y + 81 = 0$

2. Prove that the circles $(x - 4)^2 + (y + 3)^2 = r^2$, $(x + 1)^2 + (y - 9) = (13 - r)^2$ touch externally.

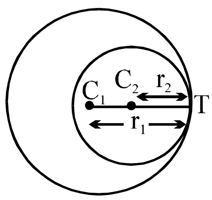
19.3 EXTERNAL AND INTERNAL CONTACTS OF CIRCLES :

If two circles with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ and radii r_1 and r_2 respectively, touch each other externally, $C_1C_2 = r_1 + r_2$. Coordinates of the point of contact are

$$A \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$$



The circles touch each other internally if $C_1C_2 = r_1 - r_2$.



Coordinates of the point of contact are

$$T \equiv \left(\frac{r_1x_2 - r_2x_1}{r_1 - r_2}, \frac{r_1y_2 - r_2y_1}{r_1 - r_2}\right)$$

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Illustration 37 : ***Find the equations of the common tangents to the circle $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$.***

Solution : The centres are $C_1(1, 3)$ an $C_2(-3, 1)$ respectively and the radii are $r_1 = \sqrt{1+9-9} = 1$ and $r_2 = \sqrt{9+1-1} = 3$.

Point I dividing $\overline{C_1C_2}$ internally in the ratio 1 : 3 is $\left(\frac{-3+3}{4}.\frac{10}{4}\right)$ or $\left(0, \frac{5}{2}\right)$.

Similarly point E dividing $\overline{C_1C_2}$ externally in the ratio 1 : 3 is (3, 4). The equation of a transverse common tangent is of the form

$y - 5/2 = m(x - 0)$ or $2mx - 2y + 5 = 0$

Since this is tangent to the first circle, its distance from the centre (1, 3) is same as its radius 1.

So $$\frac{2m(1) - 2(3) + 5}{\sqrt{4m^2 + 4}} = 1$$

or $(2m - 1)^2 = 4m^2 + 4$ or $m = -\frac{3}{4}$

This implies that the common transverse tangents are coincident (i.e., circles touch externally) and its equation is $y - \frac{5}{2} = -\frac{3x}{4}$ or $3x + 4y - 10 = 0$. The equation of a direct common tangent is of form

$y - 4 = m(x - 3)$

or $mx - y + 4 - 3m = 0$

Since this line is tangent to the circle, we get

$$\frac{m(1) - 3 + 4 - 3m}{\sqrt{m^2 + 1}} = 1$$

or $(1 - 2m)^2 = m^2 + 1$ or $(3m - 4)\, m = 0$ or $m = 0, \frac{4}{3}$

So the commontangents are

$$y - 4 = 0\,(x - 3) \text{ or } y = 4 \text{ and } y - 4 = \frac{4}{3}(x - 3) \text{ or } 4x - 4y = 0$$

REMARK

The students are advised to find the tangents without resorting to the results of elementary geometry as an exercise.

20. LOCUS PROBLEMS

Illustration 38 : ***Prove that the locus of the mid-points of chords of gradient 4 of the circle $x^2 + y^2 - 4x + 6y = 7$ is the line $x + 4y + 10 = 0$.***

Solution : The centre of the circle is C(2, – 3). Let M(h, k) be the mid-point of a chord ; then the gradient of CM is $(k + 3)/(h - 2)$. But CM is perpendicular to the chord and consequently its gradient is $-1/4$. Hence, equating the two expressions for the gradient of CM, we have $\frac{k+3}{h-2} = -\frac{1}{4}$ or $h + 4k + 10 = 0$.

Thus the locus of M(h, k) is $x + 4y + 10 = 0$.

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EXERCISE - 16

1. A variable line perpendicular to the line joining A(a, 0) and B(0, b) cuts the axes at P and Q. Find the locus of the centre of the circle on PQ as diameter.
2. The length of the tangent from a variable point P to the circle $x^2 + y^2 = 4$ is 8; find the locus of P.
3. Find the locus of points from which the lengths ofthe tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x + 6y = 8$ are equal and prove that the locus is the line of the common chord.
4. Prove that the locus of the points from which tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y + 12 = 0$ are equal is the line of the common chord AB whose length is 12/5. Prove also that the tangent of the acute angle between the tangents at A to the circles is 3/2.
5. Prove that the locus of the mid-points of the chords of contact of tangents to the circle $x^2 + y^2 = 1$ from points which lie on the line $3x + 4y + 6 = 0$ is the circle $6(x^2 + y^2) + 3x + 4y = 0$.
6. Show that the locus of a point such that the ratio of its distances from two given points is constant is a circle.
7. Show that the locus of points the tangents from which to two given circles, bear to one another a constant ratio is a circle.

❒❒

SOLVED EXAMPLES

SECTION - I

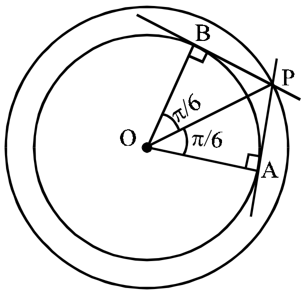
SUBJECTIVE QUESTIONS

Problem 1. Find the locus of the points of intersection of the tangents to the circle $x = r\cos\theta$, $y = r\sin\theta$ at points whose parametric angles differ by $\pi/3$.

Solution: All such points P satisfying the given condition will be equidistant from the origin O (see fig.).

Hence the locus of P will be a circle centred at the origin, having radius equal to

$$OP = \frac{r}{\cos\left(\frac{\pi}{6}\right)} = \frac{2r}{\sqrt{3}}$$



Therefore, equation of the required locus is $x^2 + y^2 = \frac{4}{3}r^2$.

Problem 2. L_1, L_2, L_3 and L_4 are the sides of a quadrilateral in that order. Find the equation of the circle circumscribing the quadrilateral without finding the vertices of the quadrilateral.

Solution: If L_1, L_2, L_3 and L_4 be the sides of the given quadrilateral, then equation of a second degree curve passing through the vertices of the triangle can be written as

$$L_1L_3 + \lambda L_2L_4 = 0 \qquad \text{.......(1)}$$

Equation (1) will represent a circle if

$$\text{coeff. of } x^2 = \text{coeff. of } y^2 \qquad \text{.......(2)}$$

and $\quad$ coeff. of $xy = 0$ $\qquad$ ....(3)

If equations (2) and (3) give the same value of λ, then the equation of the required circle can be found by putting this value of λ in equation (1).

If a unique value of λ is not obtained, it implies that the quadrilateral is not cyclic and hence equation of a circle circumscribing the quadrilateral is not possible.

Problem 3. If the curves

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and $\quad Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$

intersect at four concyclic points, prove that $\frac{(a-b)}{h} = \frac{(A-B)}{H}$.

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Solution: Equation of a curve passing through the intersection points of the given curves can be written as

$(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) + \lambda(Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C) = 0$ ....(1)

If this curve must be a circle, then

coeff. of x^2 = coeff. of y^2

i.e. $(a + \lambda A) = (b + \lambda B)$ gives $\lambda = \dfrac{b-a}{A-B}$ ....(2)

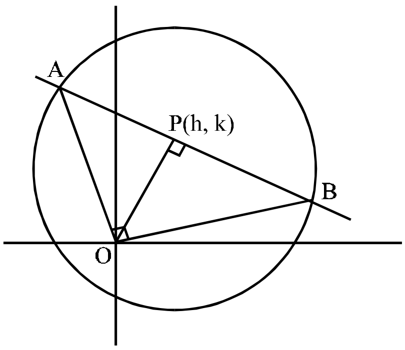
and coeff. of xy = 0

i.e. $2(h + \lambda H) = 0$ given $\lambda = -\dfrac{h}{H}$ ....(3)

Equating the two values of λ, we get the desired result.

Problem 4. Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be a given circle. Find the locus of the foot of the perpendicular drawn from the origin upon any chord of S which subtends right angle at the origin.

Solution:



AB is a variable chord such that = $\angle AOB = \dfrac{\pi}{2}$.

Let P(h, k) be the foot of the perpendicular drawn from origin upon AB. Equation of the chord AB is $y - k = \dfrac{-h}{k}(x - h)$

i.e. $hx + ky = h^2 + k^2$ ....(1)

Equation of the pair of straight lines passingh through the origin and the intersection point of the given circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ ....(2)

and the variable chord AB is

$$x^2 + y^2 + 2(gx + fy)\left(\frac{hx+ky}{h^2+k^2}\right) + c\left(\frac{hx+ky}{h^2+k^2}\right)^2 = 0 \qquad ....(3)$$

If equation (3) must represent a pair of perpendicular lines, then we have

coeff. of x^2 + coeff. of y^2 = 0

i.e. $$\left(1+\frac{2gh}{h^2+k^2}+\frac{ch^2}{(h^2+k^2)^2}\right)+\left(1+\frac{2fk}{h^2+k^2}+\frac{ck^2}{(h^2+k^2)^2}\right)=0$$

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Putting (x, y) in place of (h, k) gives the equation of the required locus as

$$x^2 + y^2 + gx + fy + \frac{c}{2} = 0$$

Problem 5. The line $Ax + By + C = 0$ cuts the circle $x^2 + y^2 + gx + fy + c = 0$ at P and Q. The line $A'x + B'y + C' = 0$ cuts the circle $x^2 + y^2 + g'x + f'y + c' = 0$ at R and S. If P, Q, R and S are concyclic, show that $\begin{vmatrix} g-g' & f-f' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$.

Solution: Equation of a circle through P and Q is

$$x^2 + y^2 + gx + fy + c + \lambda(Ax + By + C) = 0$$

i.e. $x^2 + y^2 + (g + \lambda A)x + (f + \lambda B)y + (c + \lambda C) = 0$ ....(1)

and equation of a circle through R and S is

$$x^2 + y^2 + g'x + f'y + c' + \mu(A'x + B'y + C') = 0$$

i.e. $x^2 + y^2 + (g' + \mu A')x + (f' + \mu B')y + (c' + \mu C') = 0$ ....(2)

If P, Q, R and S are concyclic points, then equations (1) and (2) must represent the same circle. Equating the ratio of the coefficients, we have $1 = \frac{g+\lambda A}{g'+\mu A'} = \frac{f+\lambda B}{f'+\mu B'} = \frac{c+\lambda C}{c'+\mu C'}$

i.e. $\lambda A - \mu A' + g - g' = 0$ ....(3)

$\lambda B - \mu B' + f - f' = 0$ ....(4)

and $\lambda C - \mu C' + c - c' = 0$ ....(5)

Eliminating λ and μ from equation (3), (4) and (5), we have $\begin{vmatrix} A & -A' & g-g' \\ B & -B' & f-f' \\ C & -C' & c-c' \end{vmatrix} = 0$

or $\begin{vmatrix} g-g' & f-f' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix}$ [interchanging rows by columns and then interchanging the second and the third row]

Aliter :

Let the given circles be

$S_1 \equiv x^2 + y^2 + gx + fy + c = 0$ ......(1)

and $S_2 \equiv x^2 + y^2 + g'x + f'y + c' = 0$ ....(2)

If S be the required circle, then according to the given condition

$Ax + By + C = 0$ is the radical axis of S_1, S

and $A'x + B'y + C' = 0$ is the radical axis of S_2, S

while $(g - g')x + (f - f')y + (c - c') = 0$ is the radical axis of S_1, S_2.

Since the radical axes of three circles taken in pairs are concurrent, therefore, we have

$\begin{vmatrix} g-g' & f-f' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$, which is the desired result.

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Problem 6. Consider a family of circles passing through the intersection point of the lines $\sqrt{3}$ (y – 1) = x – 1 and y – 1 = $\sqrt{3}$ (x – 1) and having its centre on the acute angle bisector of the given lines. Show that the common chords of each member of the family and the circle $x^2 + y^2 + 4x - 6y + 5 = 0$ are concurrent. Find the point of concurrency.

Solution: The given lines

$$\sqrt{3}(y-1) = x-1 \qquad \text{.......(1)}$$

and $$y - 1 = \sqrt{3}(x-1) \qquad \text{......(2)}$$

intersect at the point (1, 1).

Rewriting the equation of the given lines such that their constant terms are both positive, we have

$$x - \sqrt{3}y + \sqrt{3} - 1 = 0 \qquad \text{......(3)}$$

and $$-\sqrt{3}x + y + \sqrt{3} - 1 = 0 \qquad \text{......(4)}$$

Here, we have

(product of coeff.'s of x) + (product of coeff.'s of y)

$$= -\sqrt{3} - \sqrt{3} = -\text{ ve quantity}$$

which implies that the acute angle between the given lines contains the origin.

Therefore, equation of the acute angle bisector of the given lines is

$$\frac{x - \sqrt{3}y + \sqrt{3} - 1}{2} = + \frac{-\sqrt{3}x + y + \sqrt{3} - 1}{2}$$

i.e. $y = x$

Any point on the above bisector can be chosen as (a, a) and equation of any circle passing through (1, 1) and having centre at (a, a) is

$$(x-a)^2 + (y-a)^2 = (1-a)^2 + (1-a)^2$$

i.e. $$x^2 + y^2 - 2ax - 2ay + 4a - 2 = 0 \qquad \text{.......(6)}$$

The common chord of the given circle

$$x^2 + y^2 + 4x - 6y + 5 = 0 \qquad \text{.......(7)}$$

and the circle represented by equation (6) is

$$(4 + 2a)x + (2a - 6)y + (7 - 4a) = 0$$

i.e. $$(4x - 6y + 7) + 2a(x + y - 2) = 0 \qquad \text{.......(8)}$$

which represents a family of straight lines passingh through the intersection point of the lines

$$4x - 6y + 7 = 0 \qquad \text{........(9)}$$

and $$x + y - 2 = 0 \qquad \text{........(10)}$$

Solving equation (9), (10) gives the coordinates of the fixed point as $\left(\frac{1}{2}, \frac{3}{2}\right)$.

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Problem 7. Find the range of values of λ for which the variable line $3x + 4y - \lambda = 0$ lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 18x - 2y + 78 = 0$ without intercepting a chord on either circle.

Solution: The given circle

$$S_1 \equiv x^2 + y^2 - 2x - 2y + 1 = 0 \qquad ....(1)$$

has centre $C_1 \equiv (1, 1)$ and radius $r_1 = 1$

The other given circle

$$S_2 \equiv x^2 + y^2 - 18x - 2y + 78 = 0 \qquad ....(2)$$

has centre $C_2 \equiv (9, 1)$ and radius $r_2 = 2$.

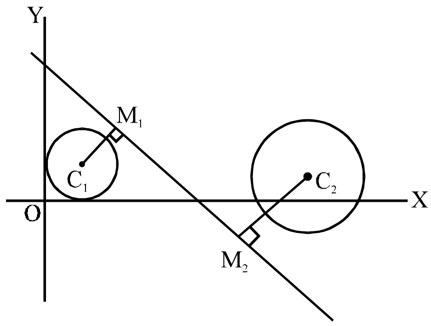
According to the required condition, we have

$$C_1M_1 \geq r_1$$

i.e. $\dfrac{|3+4-\lambda|}{\sqrt{3^2+4^2}} \geq 1$

i.e. $(\lambda - 7) \geq 5$ [$\because$ C_1 lies below the line $(7 - \lambda)$ is a –ve quantity]

i.e. $\lambda \geq 12$



and also, we have

$$C_2M_2 \geq r_2$$

i.e. $\dfrac{|27+4-\lambda|}{\sqrt{3^2+4^2}} \geq 2$

i.e. $(31 - \lambda) \geq 10$ [$\because$ C_2 lies above the line, $(31 - \lambda)$ is a +ve quantity]

i.e. $\lambda \leq 21$

Hence, the permissible values of λ are $12 \leq \lambda \leq 21$.

Problem 8. Prove that the square of the tangent that can be drawn from any point on one circle to another circle is equal to twice the product of perpendicular distance of the point from the radical axis of two circles and distance between their centres.

Solution: Let us choose the circles, as

$$S_1 \equiv x^2 + y^2 - a^2 = 0 \qquad ....(1)$$

and $S_2 \equiv (x - b)^2 + y^2 - c^2 = 0$ ....(2)

Let $P(a\cos\theta, a\sin\theta)$ be any point on circle S_1. The length of the tangent from P to circle S_2, is given by

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$PT^2 = S_2(a\cos\theta, a\sin\theta)$

$= (a\cos\theta - b)^2 + (a\sin\theta)^2 - c^2 = a^2 + b^2 - c^2 - 2ab\cos\theta$

The distance between the centres of S_1 and S_2, is

$C_1C_2 = b$

The radical axis of S_1 and S_2, is

$2bx - a^2 - b^2 + c^2 = 0$ [equation (1) – equation (2)]

The perpendicular distance of P from the radical axis, is

$$PM = \frac{|2b(a\cos\theta)-a^2-b^2+c^2|}{2b}$$

Now, we have

$2.\ PM.\ C_1C_2 = 2b.\ \dfrac{|2ab\cos\theta-a^2-b^2+c^2|}{2b} = |a^2 + b^2 - c^2 - 2ab\cos\theta| = PT^2$ which proves the desired result.

Problem 9. If tangents drawn from two given points to a variable circle, are of given lengths. Prove that the circle passes through two fixed points, real or imaginary.

Solution: Let us choose the given points as O(0, 0) and A(a, 0) and the equation of a variable circle as

$S(x, y) \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ ....(1)

The length of the tangents from the two given point O and A to S, are given by

$l^2 = S(0, 0) = c$ ....(2)

and $m^2 = S(a, 0) = a^2 + 2ag + c$ ....(3)

where l and m are given constants.

From equations (2) and (3), we have

$$c = l^2 \text{ and } 2g = \frac{m^2-l^2-a^2}{a}$$

Putting in equation (1), we have

$$S(x, y) = x^2 + y^2 + \left(\frac{m^2-l^2-a^2}{a}\right)x + l^2 + 2fy = 0$$

which represents the equation of a family of circles all passing through the intersection points of

$$x^2 + y^2 + \left(\frac{m^2-l^2-a^2}{a}\right)x + l^2 = 0 \text{ and } y = 0$$

Solving, we have

$ax^2 + (m^2 - l^2 - a^2)x + al^2 = 0$

which will give two values of x, real or imaginary.

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Problem 10. $x + y - 6 = 0$, $2x + y - 4 = 0$ and $x + 2y - 5 = 0$ are the sides of a triangle. Find the equation of the circumcircle of the triangle without finding the vertices of the triangle.

Solution: If L_1, L_2 and L_3 be the sides of given triangle, then equation of a second degree curve passing through the vertices of the triangle can be written as

$$L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$$

i.e. $(x + y - 6)(2x + y - 4) + \lambda(2x + y - 4)(x + 2y - 5) + \mu(x + 2y - 5)(x + y - 6) = 0$

i.e. $(2x^2 + y^2 + 3xy - 16x - 10y + 24)$

$+ \lambda(2x^2 + 2y^2 + 5xy - 14x - 13y + 20)$

$+ \mu(x^2 + 2y^2 + 3xy - 11x - 17y + 30) = 0$

i.e. $(2 + 2\lambda + \mu)x^2 + (1 + 2\lambda + 2\mu)y^2 + (3 + 5\lambda + 3\mu)xy - (16 + 14\lambda + 11\mu)x$

$-(10 + 13\lambda + 17\mu)y + (24 + 20\lambda + 30\mu) = 0$ ....(1)

Equation (1) will represent a circle if

coeff. of x^2 = coeff. of y^2

i.e. $2 + 2\lambda + \mu = 1 + 2\lambda + 2\mu$ gives $\mu = 1$ ....(2)

and coeff. of $xy = 0$

i.e. $3 + 5\lambda + 3\mu = 0$ given $\lambda = -\dfrac{6}{5}$ ....(3)

Hence, the equation of the required circle is

$$\frac{3}{5}(x^2 + y^2) - \frac{51}{5}x - \frac{57}{5}y + \frac{150}{5} = 0$$

i.e. $x^2 + y^2 - 17x - 19y + 50 = 0$.

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SECTION - II

SINGLE AND MULTIPLE CHOICE PROBLEMS

Problem 1. Two circles with radii 'r_1' and 'r_2' and $r_1 > r_2$ touch each other externally. If 'θ' be the angle between the direct common tangents, then

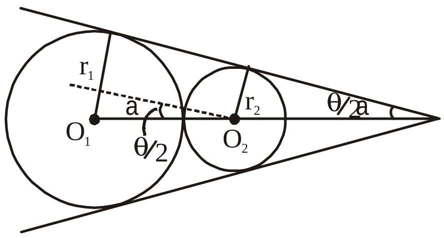
(a) $\theta = \sin^{-1}\left(\frac{r_1+r_2}{r_1-r_2}\right)$

(b) $\theta = 2\sin^{-1}\left(\frac{r_1-r_2}{r_1+r_2}\right)$

(c) $\theta = \sin^{-1}\left(\frac{r_1-r_2}{r_1+r_2}\right)$

(d) none of these

Solution: **Ans. (b)**



$$\sin(\theta/2) = \frac{r_1-r_2}{r_1+r_2} \quad \Rightarrow \quad \theta = 2\sin^{-1}\left(\frac{r_1-r_2}{r_1+r_2}\right)$$

Hence (b) is correct

Problem 2. If the curves $ax^2 + 4xy + 2y^2 + x + y + 5 = 0$ and $ax^2 + 6xy + 5y^2 + 2x + 3y + 8 = 0$ intersect at four concyclic points then the value of a is -

(a) 4

(b) – 4

(c) 6

(d) – 6

Solution: **Ans. (b)**

Any second degree curve passing through the intersection of the given curves is

$$ax^2 + 4xy + 2y^2 + x + y + 5 + \lambda(ax^2 + 6xy + 5y^2 + 2x + 3y + 8) = 0$$

If it is a circle, then coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0

$$a(1 + \lambda) = 2 + 5\lambda \text{ and } 4 + 6\lambda = 0$$

$$\Rightarrow \quad a = \frac{2+5\lambda}{1+\lambda} \quad \text{and} \quad \lambda = -\frac{2}{3}$$

$$\Rightarrow \quad a = \frac{2-\frac{10}{3}}{1-\frac{2}{3}} = -4.$$

Hence (b) is correct answer.

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Problem 3. Equation of chord AB of circle $x^2 + y^2 = 2$ passing through P(2, 2) such that PB/PA = 3, is given by -

(a) $x = 3y$ (b) $x = y$

(c) $y - 2 = \sqrt{3}\ (x - 2)$ (d) none of these

Solution: **Ans. (b)**

Any line passing through (2, 2) will be of the form $\dfrac{y-2}{\sin\theta} = \dfrac{x-2}{\cos\theta} = r$

When this line cuts the circle $x^2 + y^2 = 2$, $(2 + \gamma\cos\theta)^2 + (2 + \gamma\sin\theta)^2 = 2$

$\Rightarrow \quad r^2 + 4(\sin\theta + \cos\theta)\, r + 6 = 0$

$\dfrac{PB}{PA} = \dfrac{r_2}{r_1}$, now if $r_1 = a$, $r_2 = 3a$,

then $\quad 4a = -4(\sin\theta + \cos\theta)$, $3a^2 = 6$

$\Rightarrow \quad \sin 2\theta = 1$

$\Rightarrow \quad \theta = \pi/4.$

So required chord will be $y - 2 = 1\,(x - 2)$

$\Rightarrow \quad y = x.$

Problem 4. Equation of a circle $S(x, y) = 0$, $S(2, 3) = 16$ which touches the line $3x + 4y - 7 = 0$ at (1, 1) is given by

(a) $x^2 + y^2 + x + 2y - 5 = 0$ (b) $x^2 + y^2 + 2x + 2y - 6 = 0$

(c) $x^2 + y^2 + 4x - 6y = 0$ (d) none of these

Solution: **Ans. (a)**

Any circle which touches $3x + 4y - 7 = 0$ at (1, 1) will be of the form

$S(x, y) \equiv (x - 1)^2 + (y - 1)^2 + \lambda\,(3x + 4y - 7) = 0$

Since $\quad S(2, 3) = 16$

$\Rightarrow \quad \lambda = 1$, so required circle will be

$x^2 + y^2 + x + 2y - 5 = 0.$

Hence (a) is the correct answer.

Problem 5. If P(2, 8) is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$ which neither touches nor intersects the axes, then set for p is -

(a) $p < -1$ (b) $P < -4$

(c) $p > 96$ (d) ϕ

Solution: **Ans. (d)**

For internal point p(2, 8),

$4 + 64 - 4 + 32 - p < 0 \Rightarrow p > 96$ and x intercept $= 2\sqrt{1+p}$ therefore $1 + p < 0$

$\Rightarrow \quad p < -1$ and y intercept $= 2\sqrt{4+p}$

$\Rightarrow \quad p < -4$

Hence (d) is the correct answer.

Problem 6. The common chord of $x^2 + y^2 - 4x - 4y = 0$ and $x^2 + y^2 = 16$ subtends at the origin an angle equal to -

(a) $\pi/6$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

Solution: **Ans. (d)**

The equation of the common chord of the circles $x^2 + y^2 - 4x - 4y = 0$ and $x^2 + y^2 = 16$ is $x + y = 4$ which meets the circle $x^2 + y^2 = 16$ at points A(4, 0) and B(0, 4). Obviously $OA \perp OB$. Hence the common chord AB makes a right angle at the centre of the circle $x^2 + y^2 = 16$.

Hence (d) is the correct answer.

Problem 7. The number of common tangents that can be drawn to the circle

$$x^2 + y^2 - 4x - 6y - 3 = 0 \text{ and } x^2 + y^2 + 2x + 2y + 1 = 0 \text{ is}$$

(a) 1 (b) 2

(c) 3 (d) 4

Solution: **Ans. (c)**

The two circles are

$$x^2 + y^2 - 4x - 6y - 3 = 0 \text{ and } x^2 + y^2 + 2x + 2y + 1 = 0$$

Centre : $C_1 \equiv (2, 3)$, $C_2 \equiv (-1, -1)$ radii : $r_1 = 4$, $r_2 = 1$

We have $C_1 C_2 = 5 = r_1 + r_2$, therefore there are 3 common tangents to the given circles. Hence (C) is the correct answer.

Problem 8. The line $y = mx + c$ intersects the circle $x^2 + y^2 = r^2$ at two real distinct points if

(a) $-r\sqrt{1+m^2} < c \le 0$ (b) $0 \le c < r\sqrt{1+m^2}$

(c) $-c\sqrt{1-m^2} < r$ (d) $r < c\sqrt{1+m^2}$

Solution: **Ans. (b)**

The x-coordinates of the points of intersection of the line $y = mx + c$ and the circle $x^2 + y^2 = r^2$ are given by $x^2 + (mx + c)^2 = r^2$

or $(1 + m^2)x^2 + 2mcx + c^2 - r^2 = 0$ ...(i)

which, being quadratic in x, gives two values of x and hence two points of intersection. These points will be real and distinct if the discriminant of (i) is positive i.e., $4m^2c^2 - 4(1 + m^2)(c^2 - r^2) > 0$

Problem 9. Tangents are drawn to the circle $x^2 + y^2 = 50$ from a point 'P' lying on the x-axis. These tangents meet the y-axis at points 'P_1' and 'P_2'. Possible coordinates of 'P' so that area of triangle PP_1P_2 is minimum, is/are

(a) (10, 0) (b) $(10\sqrt{2}, 0)$

(c) (–10, 0) (d) $(-10\sqrt{2}, 0)$

Solution: **Ans. (a), (c)** $OP = 5\sqrt{2}\sec\theta$,

$OP_1 = 5\sqrt{2}\,\text{cosec}\theta$

$\Delta PP_1P_2 = \dfrac{100}{\sin 2\theta}$

$(\Delta PP_1, P_2)_{min} = 100$

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$\Rightarrow \quad \theta = \dfrac{\pi}{4}$

$\Rightarrow \quad OP = 10$

$\Rightarrow \quad P = (10, 0), (-10, 0)$

Hence (a), (c) are correct

Problem 10. If (a, 0) is a point on a diameter of the circle $x^2 + y^2 = 4$, then $x^2 - 4x - a^2 = 0$ has

(a) Exactly one real root in (– 1, 0)
(b) Exactly one real root in (2, 5)
(c) Distinct roots greater than-1
(d) Distinct roots less than 5

Solution: **Ans. (a), (b), (c) and (d)**

Since (a, 0) is a point on the diameter of the circle $x^2 + y^2 = 4$,

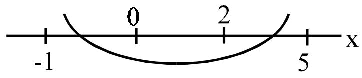
So maximum value of a^2 is 4

Let $\quad f(x) = x^2 - 4x - a^2$

clearly $\quad f(-1) = 5 - a^2$ is $1 > 0$

$f(2) = -(a^2 + 4) < 0$

$f(0) = -a^2 < 0$ and $f(5) = 5 - a^2 > 0$



so graph of f(x) will be as shown

Hence (a), (b), (c), (d) are the correct answer.

Problem 11. The equation (s) of the tangent at the point (0, 0) to the circle, making intercepts of length 2a and 2b units on the coordinate axes, is (are) -

(a) $ax + by = 0$
(b) $ax - by = 0$
(c) $x = y$
(d) None of these

Solution: **Ans. (a), (b)**

Equation of circle passing through origin and cutting off intercepts 2a and 2b units on the coordinate axes is $x^2 + y^2 \pm 2ax \pm 2by = 0$

Hence (a), (b) are correct answers.

Problem 12. The locus of the point of intersection of the tangents at the extremities of a chord of the circle $x^2 + y^2 = a^2$ which touches the circle $x^2 + y^2 - 2ax = 0$ passed through the point

(a) (a/2, 0)
(b) (0, a/2)
(c) (0, a)
(d) (a, 0)

Solution: **Ans. (a), (c)**

Let P(h, k) be the point of intersection of the tangents at the extremities of the chord AB of the circle $x^2 + y^2 = a^2$. Since AB is the chord of contact of the tangents from P to this circle, its equation is $hx + ky = a^2$. If this line touches the circle $x^2 + y^2 - 2ax = 0$, then

$$\frac{h \cdot a + k \cdot 0 - a^2}{\sqrt{h^2 + k^2}} = \pm a \quad \Rightarrow (h - a)^2 = h^2 + k^2$$

Therefore, the locus of (h, k) is $(x - a)^2 = x^2 + y^2$, or $y^2 = a(a - 2x)$, which passes through the points given in (a) and (c)

Problem 13. Equation of the circle with centre (4, 3) and touching the circle $x^2 + y^2 = 1$ is

(a) $x^2 + y^2 - 8x - 6y + 9 = 0$ (b) $x^2 + y^2 + 8x + 6y - 11 = 0$

(c) $x^2 + y^2 - 8x - 6y - 11 = 0$ (d) $x^2 + y^2 + 8x + 6y - 9 = 0$

Solution: **Ans. (a), (c)**

Let equation of the required circle be $(x - 4)^2 + (y - 3)^2 = r^2$ ...(i)

If the circle (i) touches the circle $x^2 + y^2 = 1$, the distance between the centres (4, 3) and (0, 0) of these circles is equal to the sum or difference of their radii, r and 1.

$$\Rightarrow \quad \sqrt{4^2 + 3^2} = 1 \pm r \quad \Rightarrow \quad r \pm 1 = 5$$

$\Rightarrow$ r = 4 or 6 so that the equations of the required circles from (i)

are $x^2 + y^2 - 8x - 6y + 9 = 0$ and $x^2 + y^2 - 8x - 6y - 11 = 0$.

Problem 14. An equation of a circle which touches the y-axis at (0, 2) and cuts off an intercept 3 from the x-axis is

(a) $x^2 + y^2 + 4x - 5y + 4 = 0$ (b) $x^2 + y^2 + 5x - 4y + 4 = 0$

(c) $x^2 + y^2 - 5x - 4y + 4 = 0$ (d) $x^2 + y^2 - 5x + 4y + 4 = 0$

Solution: **Ans. (b), (c)**

As the required circle touches the y-axis at (0, 2), let its equation be

$$(x - \alpha)^2 + (y - 2)^2 = \alpha^2 \quad \text{or} \quad x^2 + y^2 - 2\alpha x - 4y + 4 = 0$$

This circle meets the x-axis at the points where $x^2 - 2\alpha x + 4 = 0$, which gives two values of x, say x_1, x_2 with $x_1 < x_2$, such that $x_1 + x_2 = 2\alpha$ and $x_1 x_2 = 4$.

Now, we are given that $|x_1 - x_2| = 3$.

$$\therefore \; (x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2 = 9$$

$$\Rightarrow \; 4\alpha^2 - 16 = 9 \; \Rightarrow \; \alpha^2 = \frac{25}{4} \Rightarrow \; \alpha = \pm\frac{5}{2}$$

Hence the equation of the required circle is $x^2 + y^2 \pm 5x - 4y + 4 = 0$.

Problem 15. Two circles $x^2 + y^2 + px + py - 7 = 0$ and $x^2 + y^2 - 10x + 2py + 1 = 0$ will cut orthogonally if the value of p is

(a) –2 (b) –3

(c) 2 (d) 3

Solution: **Ans. (c), (d)**

The given circles will cut orthogonally if

$$2\left(\frac{p}{2}\right)\left(\frac{-10}{2}\right) + 2\left(\frac{p}{2}\right)\left(\frac{2p}{2}\right) = -7 + 1$$

or $p^2 - 5p + 6 = 0$ if p = 2 or 3.

SECTION - III

COMPREHENSION TYPE PROBLEMS

Write up – I

Consider the family of circles $x^2 + y^2 = r^2$, 2< r< 5. If in the first quadrant the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the coordinate axes at A and B

1. Find the coordinate of A

(a) $\left(\frac{2}{\cos\theta},0\right)$ (b) $\left(\frac{3}{\cos\theta},0\right)$

(c) $\left(\frac{5}{\cos\theta},0\right)$ (d) $\left(\frac{7}{\cos\theta},0\right)$

2. Find the coordinate of B

(a) $\left(0,\frac{2}{\sin\theta}\right)$ (b) $\left(0,\frac{3}{\sin\theta}\right)$

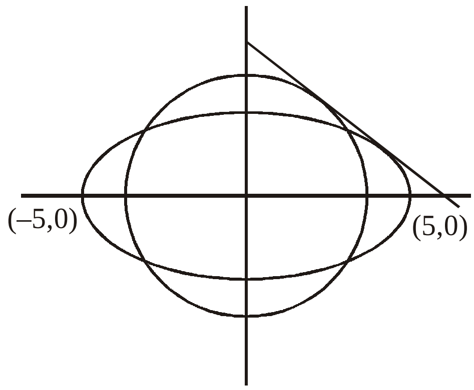
(c) $\left(0,\frac{5}{\sin\theta}\right)$ (d) $\left(0,\frac{7}{\sin\theta}\right)$

3. Minimum area of OAB

(a) 10 sq.units (b) 30 sq.units

(c) 25 sq.units (d) 15 sq.units

Solution :



Any point on ellipse is $P(5\cos\theta, 2\sin\theta)$ tangent at P

$$4\cdot 5\cos\theta\cdot x + 25\cdot 2\sin\theta y = 100$$

$$2\cos\theta x + 5\sin\theta y = 10$$

1. If y = 0, $x = \left(\frac{5}{\cos\theta}, 0\right)$, $A = \left(\frac{5}{\cos\theta}, 0\right)$

2. If x = 0, $y = \frac{2}{\sin\theta} \Rightarrow B \equiv \left(0, \frac{2}{\sin\theta}\right)$

3. Area $A = \frac{1}{2} OA \cdot OB = \frac{1}{2} \cdot \frac{5}{\cos\theta} \cdot \frac{2}{\sin\theta} = \frac{10}{\sin 2\theta}$

A is minimum when $\sin 2\theta = 1$

$A_{(min)} = 10$

Write up – II

A point P 'P' moves in a plane such that $\frac{PA}{PB} = \lambda$, where $\lambda \in (0, 1)$ is a constant and A(0, 0), B(a, 0) are fixed points where AB = a

4. Then the locus of P is

(a) a circle passing through A
(b) circle passing through B
(c) a straight line if $\lambda > 1$
(d) circle not passing through A and B

5. The locus of P is a circle whose diameter is

(a) $\frac{a\lambda}{1-\lambda^2}$
(b) $\frac{a\lambda}{2(1-\lambda^2)}$
(c) $\frac{2a\lambda}{1-\lambda^2}$
(d) None of these

6. The locus of 'P' will have

(a) the point A as it's interior point
(b) the point B as it's interior point
(c) the points A and B both as it's interior points
(d) none of these

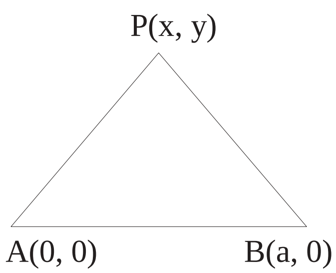
Solution :

4. $PA^2 = \lambda^2 PB^2$ & AB = a

$x^2 + y^2 = \lambda^2 [(x - a)^2 + y^2]$

$(1 - \lambda^2) x^2 + (1 - \lambda^2) y^2 + 2\lambda^2 ax - \lambda^2 a^2 = 0$

$x^2 + y^2 + \frac{2\lambda^2}{1-\lambda^2} ax - \frac{\lambda^2 a^2}{1-\lambda^2} = 0$



5. $r = \sqrt{\left(\frac{\lambda^2 a}{1-\lambda^2}\right)^2 + \frac{\lambda^2 a^2}{1-\lambda^2}} = \frac{\lambda a}{1-\lambda^2}\sqrt{\lambda^2 + (1-\lambda^2)} = \frac{\lambda a}{1-\lambda^2}$

diameter = 2r = $\frac{2\lambda a}{1-\lambda^2}$

6. Clearly A(0, 0) is interior point of locus of P.

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MATCHING TYPE PROBLEM

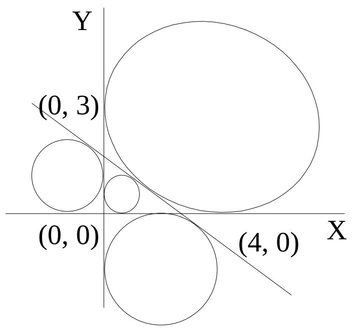
Problem :

Let C_1 and C_2 be two circles whose equations are $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 2x = 0$. P (λ, λ) is a variable point. Then match the following

Column I	***Column II***
(a) Sum of all the radii of the circles touching the coordinate axes and the line $3x + 4y = 12$ is	(P) 1
(b) If the total number of integral points in the interior of the circle $x^2 + y^2 = 9$ are equal to $4n + 1$, then n is equal to	(Q) 6
(c) The sum of the square of the length of the chords intercepted by the line $x - y = 8n$, $n \in N$ on the circle $x^2 + y^2 = 36$, is	(R) 12
(d) A circle of radius unity touches the co–ordinate axes in the first quadrant, then sum of all the radii of the circles touching this circle and co–ordinate axes is	(S) 25

Sol. ***Ans. (a) – (R), (b) – (S), (c) – (P), (d) – (Q)***

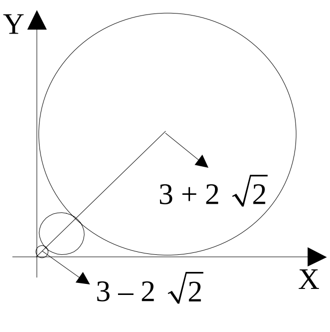
(a) Required sum = 2 [5 + 1] = 12



(b) Total number of integral points are 25.

(c) $\left|\frac{8n}{\sqrt{2}}\right|^2 < 36 \Rightarrow 64n^2 < 72 \quad \therefore n = 1.$

(d) Sum = 6



ASSERTION-REASON TYPE PROBLEMS

In the following questions containing two statements viz. Assertion (A) & Reason (R). To choose the correct answer.

Mark (a) if both A and R are correct & R is the correct explanation for A.

Mark (b) if both A & R are correct but R is not correct explanation for A

Mark (c) if A is true but R is false

Mark (d) if A is false but R is true

1. **A :** ($n \geq 3$) for n circles, the value of n for which the number of radical axes is equal to the number of radical centres, is 5.

   R : If no two of n circles are concentric, no three of the centres are collinear then number of possible radical centres is ${}^nC_3$.

2. **A :** The circle $x^2+y^2+2ax+c=0, x^2+y^2+2by+c=0$ touch if $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c}$.

   R : Two circles with centres C_1, C_2 and radii r_1, r_2 touch each other if $r_1 \pm r_2 = C_1C_2$.

3. **A :** The common tangents of the circles $x^2+y^2+2x=0, x^2+y^2-6x=0$ form an equilateral triangle.

   R : The given circles touch each other externally.

Solution :

1. No. of radical axes = no. of radical centres.

   ${}^nC_2 = {}^nC_3 \Rightarrow$ n = 5 both are true.

2. Two circles touch each other

   $C_1C_2 = r_1 \pm r_2$

   $$\Rightarrow c^2=\left(a^2-1\right)\left(b^2-c\right) \Rightarrow a^2b^2=\left(a^2+b^2\right)c \Rightarrow \frac{1}{c}=\frac{1}{a^2}+\frac{1}{b^2}$$

   Both are true and R is correct reason of A.

3. For $x^2+y^2+2x=0, C_1\left(-1,0\right), r_1=1,$ for $x^2+y^2-6x=0, C_2(3,0), r_2=3$.

   $\Rightarrow$ Point of contact of circle

   $E \equiv \left(\frac{3+3}{1-3}, 0\right)$ = (– 3, 0), angle between the tangents from (– 3, 0) to

   $x^2 + y^2 + 2x = 0$ is $\theta=2\tan^{-1}\left(\frac{1}{\sqrt{9+0-6}}\right)=\frac{\pi}{3}$ and $C_1C_2=r_1+r_2$.

   Therefore the common tangents form an equilateral triangle and they touch each other externally.

   The assertion A is correct and reason is also correct, but R is not the correct explanation of A.

❑❑

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ASSIGNMENTS

SECTION - I

SUBJECTIVE QUESTIONS

LEVEL - I

1. The vertices of a triangle are (2, –1), (2, 3) and (4, – 1); prove that the centre and radius of the circumcircle are (3, 1) and $\sqrt{5}$.

2. Find the coordinates of the centres of the two circles, each of radius 4, passing through the points (2, 4) and (6, 8).

3. P is any point on the circle whose centre is A (a, b) and which passes through the origin. Prove that the locus of the centroid of the triangle OAP is

$$3(x^2 + y^2) – 4ax – 4by + a^2 + b^2 = 0.$$

4. Find the centre and radius of each of the circles :

   (i) $(x – a)(x – b) + (y – c)(y – d) = 0;$

   (ii) $(x + y + a)^2 + (x – y – a)^2 = 8a^2.$

5. The square of the distance of a variable point P from the origin is 4 times the distance of P from the line x = 1. Prove that the locus of P is either the point-circle (2, 0) or the circle $(x + 2)^2 + y^2 = 8$.

6. Find the equations to the circle which touches the y-axis at the point (0, 3) and which has intercept 8 on the positive x-axis.

7. Show that the line 4x – 7y + 28 does not intersect the circle $x^2 + y^2 – 6x – 8y = 0$.

8. Find the equation of the common chord of the circles $x^2 + y^2 + 2x – 4y – 20 = 0$ and $3x^2 + 3y^2 + 5x – 11y – 70 = 0$. Verify that this chord is perpendicular to the line joining the centres of the circles.

9. Find the equations of the circles which have radius $\sqrt{13}$ and which touch the line 2x – 3y + 1 = 0 at (1, 1).

10. A is (3, 7), B is (– 1, 5) and P is a variable point such that $AP^2 + BP^2 = 82$. Prove that the locus of P is a circle of radius 6.

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LEVEL - II

1. Prove that for all values of the constant p and q, the circle $(x - a)(x - a + p) + (y - b)(y - b + q) = r^2$ bisects the circumference of circle $(x - a)^2 + (y - b)^2 = r^2$.

2. Show that the circle of which line joining points $(at^2, 2at)$ and $(a/t^2, 2a/t)$ as is a diameter, touches $x + a = 0$ for all values of t.

3. Show that the equation to the circle of which (x_1, y_1) and (x_2, y_2) are the ends of a chord of a segment containing an angle θ at the centre is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) \pm \cot\theta[(x - x_1)(y - y_1) - (x - x_2)(y - y_2)] = 0.$$

4. Prove that the length of the common chord of the circles

$$(x - a)^2 + (y - b)^2 = c^2 \text{ and } (x - b)^2 + (y - a)^2 = c^2 \text{ is } \sqrt{4c^2 - 2(a-b)^2}.$$

Hence find the condition that the two circles touch each other.

5. A straight line segment of given length 2 is made to slide so that its extremities always lie on the axes. Show that the locus of its middle point is a circle of radius 1 whose centre is the origin.

6. Find the equations of common tangents to the circles

$$x^2 + y^2 + 4x + 2y - 4 = 0 \text{ and } x^2 + y^2 - 4x - 2y + 4 = 0.$$

7. Prove that the locus of points, the tangents from which to two given circles are in a given ratio is a circle coaxial with the given circles.

8. Prove that the equation of the circle orthogonal to the three circles

$$x^2 + y^2 + 2d_1 x + 2e_1 y + f_1 = 0$$

$$x^2 + y^2 + 2d_2 x + 2e_2 y + f_2 = 0$$

$$x^2 + y^2 + 2d_3 x + 2e_3 y + f_3 = 0$$

is $$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ -f_1 & d_1 & e_1 & -1 \\ -f_2 & d_2 & e_2 & -1 \\ -f_3 & d_3 & c_3 & -1 \end{vmatrix} = 0.$$

9. Find the equations to the straight lines joining the origin to the points of intersection of

$$x^2 + y^2 - 4x - 2y = 4 \text{ and } x^2 + y^2 - 2x - 4y - 4 = 0.$$

10. The straight line $lx + my - 1 = 0$ meets the lines

$$ax^2 + 2hxy + by^2 = 0$$

in the points P and Q; show that the equation of the circle described on PQ as diameter is

$$(x^2 + y^2)(am^2 - 2hlm + bl^2) - 2x(bl - hm) - 2y(am - hl) + a + b = 0.$$

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LEVEL - III

1. A variable line from (3, 4) meets the line $4x + 3y = 12$ at P ; a point Q is taken in AP such that AP. AQ = c^2. Prove that the locus of Q is the circle.

$$12(x - 3)^2 + 12(y - 4)^2 + c^2 (4x + 3y - 24) = 0.$$

2. P is a variable point on the circle whose centre is C (1, 2) and which passes through the origin. Prove that the locus of the centroid of triangle OCP is $3(x^2 + y^2) - 4x - 8y + 5 = 0$.

3. Find the point on the circle $x^2 + y^2 - 6x - 8y + \frac{125}{9} = 0$ which is

   (i) nearest the origin and

   (ii) farthest from (– 9, – 5).

4. Prove that the greatest and the least distance from the origin to points on the circle $(x - a)^2 + (y - b)^2 = r^2$ are given by the numerical values of the roots of the equation

$$R^2 - 2R\sqrt{(a^2+b^2)} + a^2 + b^2 - r^2 = 0.$$

5. Prove that the equation of the circle circumscribing the triangle formed by the lines

$$Ax + By + C = 0,\ A' x + B'y + C' = 0\ A'' x + B''y + C'' = 0$$

is
$$\begin{vmatrix} \frac{A^2+B^2}{Ax+By+C} & A & B \\ \frac{A'^2+B'^2}{A'x+B'y+C'} & A' & B' \\ \frac{A''^2+B''^2}{A''x+B''y+C''} & A'' & B'' \end{vmatrix} = 0.$$

6. Show that if $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ be the equations of the three circles of which each two cut orthogonally, the equation

$$\ell_1 S_1 + \ell_2 S_2 + \ell_3 S_3 = 0$$

represents a real circle except in certain cases where it represents a straight line.

7. Prove that if $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, $S_4 = 0$ be four circles of which each pair is orthogonal, their equations being in the form in which S denotes the square of the tangent from any point, then the condition so that

$$\lambda S_1 + \mu S_2 + nS_3 + rS_4 = 0 \text{ and } \lambda' S_1 + \mu' S_2 + n' S_3 + r' S_4 = 0$$

should be orthogonal if

$$\lambda\lambda' r_1^2 + \mu\mu' r_2^2 + nnr_3^2 + rrr_4^2 = 0$$

where r_1, r_2, r_3, r_4 are the radii of the circle.

8. From a fixed point (x, y) perpendiculars are drawn to the straight lines

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Show that the equation of the circle circumscribing the quadrilateral so formed is

$$(ab - h^2) \{x(x - x) + y (y - h)\} - (hf - bg)(x - x) - (gh - af)(y - h) = 0.$$

9. Prove that if four points P, Q, R, S be taken, and the square of the tangent from P to the circle on QR as diameter be denoted by (P, QR), then

$$(P, RS) - (P, QS) - (Q, RS) + (Q, PR) = 0.$$

10. Prove that the two circles, which pass through the two points (0, a) and (0, –a) and touch the straight line $y = mx + c$, will cut orthogonally if $c^2 = a^2 (2 + m^2)$. ❑❑

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SECTION - II

SINGLE CHOICE QUESTIONS

1. The circle $x^2 + y^2 + 2a_1x + c = 0$ lies completely inside the circle $x^2 + y^2 + 2a_2x + c = 0$ then-
(a) $a_1a_2 > 0, c < 0$
(b) $a_1a_2 > 0, c > 0$
(c) $a_1a_2 < 0, c < 0$
(d) $a_1a_2 < 0, c > 0$

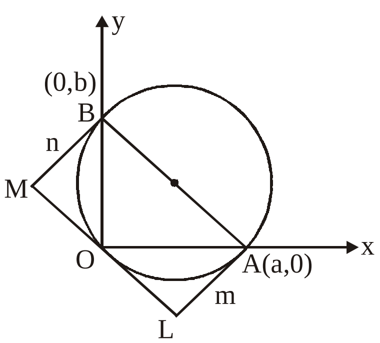
2. Angle between tangents drawn to $x^2 + y^2 - 2x - 4y + 1 = 0$ at the points where it is cut by the line $y = 2x + c$, is $\frac{\pi}{2}$ then
(a) $|c| = \sqrt{5}$
(b) $|c| = 2\sqrt{5}$
(c) $|c| = \sqrt{10}$
(d) $|c| = 2\sqrt{10}$

3. A circle touches the lines $y = \frac{x}{\sqrt{3}}$, $y = x\sqrt{3}$ and has unit radius. If the centre of this circle lies in the first quadrant then possible equation of this circle is -
(a) $x^2 + y^2 - 2x(\sqrt{3} + 1) - 2y(\sqrt{3} + 1) + 8 + 4\sqrt{3} = 0$
(b) $x^2 + y^2 - 2x(1 + \sqrt{3}) - 2y(1 + \sqrt{3}) + 5 + 4\sqrt{3} = 0$
(c) $x^2 + y^2 - 2x(1 + \sqrt{3}) - 2y(1 + \sqrt{3}) + 7 + 4\sqrt{3} = 0$
(d) $x^2 + y^2 - 2x(1 + \sqrt{3}) - 2y(1 + \sqrt{3}) + 6 + 4\sqrt{3} = 0$

4. If $2x^2 + \ell xy + 2y^2 + (\ell - 4)x + 6y - 5 = 0$ is the equation of a circle then its radius is -
(a) $3\sqrt{2}$
(b) $2\sqrt{3}$
(c) $2\sqrt{2}$
(d) none of these

5. The equation $x^2 + y^2 - 2x + 4y + 5 = 0$ represents -
(a) a point
(b) a pair of straight lines
(c) a circle of nonzero radius
(d) none of these

6. A line meets the coordinate axes in A and B. A circle is circumscribed about the triangle OAB. If m and n are the distances of the tangent to the circle at the origin from the points A and B respectively, the diameter of the circle is
(a) $m(m + n)$
(b) $m + n$
(c) $n(m + n)$
(d) $(1/2)(m + n)$



7. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = k^2$ orthogonally, equation of the locus of its centre is

(a) $2ax + 2by = a^2 + b^2 + k^2$
(b) $ax + by = a^2 + b^2 + k^2$
(c) $x^2 + y^2 + 2ax + 2by + k^2 = 0$
(d) $x^2 + y^2 - 2ax - 2by + a^2 + b^2 - k^2 = 0$

8. Equation of the circle which passes through the origin, has its centre on the line x + y = 4 and cuts the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally, is

(a) $x^2 + y^2 - 2x - 6y = 0$
(b) $x^2 + y^2 - 6x - 3y = 0$
(c) $x^2 + y^2 - 4x - 4y = 0$
(d) none of these

9. If O is the origin and OP, OQ are distinct tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then circumcentre of the triangle OPQ is

(a) (–g, –f)
(b) (g, f)
(c) (–f, –g)
(d) none of these

10. The circle passing through the distinct points (1, t), (t, 1) and (t, t) for all values of t, passes through the point

(a) (1, 1)
(b) (–1, –1)
(c) (1, –1)
(d) (–1, 1)

11. If OA and OB are the tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, and C is the centre of the circle, the area of the quadrilateral OACB is

(a) $\frac{1}{2}\sqrt{c(g^2 + f^2 - c)}$
(b) $\sqrt{c(g^2 + f^2 - c)}$
(c) $c\sqrt{g^2 + f^2 - c}$
(d) $\frac{\sqrt{g^2 + f^2 - c}}{c}$

12. If a line segment AM = a moves in the plane XOY remaining parallel to OX so that the left end point A slides along the circle $x^2 + y^2 = a^2$, the locus of M is

(a) $x^2 + y^2 = 4a^2$
(b) $x^2 + y^2 = 2ax$
(c) $x^2 + y^2 = 2ay$
(d) $x^2 + y^2 - 2ax - 2ay = 0$

13. The line 9x + y – 28 = 0 is the chord of contact of the point P(h, k) with respect to the circle $2x^2 + y^2 - 3x + 5y - 7 = 0$, for

(a) P(3, –1)
(b) P(3, 1)
(c) P(–3, 1)
(d) no position of P

14. The locus of the poles of the line lx + my + n = 0 with respect to circles which touch the x-axis at the origin is

(a) (lx + my) y = nx
(b) (mx – ly) y = nx
(c) (lx – my) x = ny
(d) (lx – my) y = nx

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15. C_1 is a circle with centre at the origin and radius equal to r and C_2 is a circle with centre at (3r, 0) and radius equal to 2r. The number of common tangents that can be drawn to the two circles are

(a) 1 (b) 2
(c) 3 (d) 4

16. An equilateral triangle is inscribed in the circle $x^2 + y^2 = a^2$ with vertex at (a, 0). The equation of the side opposite to this vertex is

(a) $2x - a = 0$ (b) $x + a = 0$
(c) $2x + a = 0$ (d) $3x - 2a = 0$

17. A line is drawn through the point P(3, 11) to cut the circle $x^2 + y^2 = 9$ at A and B. Then PA . PB is equal to

(a) 9 (b) 121
(c) 205 (d) 139

18. Two rods of lengths a and b slide along the x-axis and y-axis respectively in such a manner that their ends are concyclic. The locus of the centre of the circle passing through the end points is

(a) $4(x^2 + y^2) = a^2 + b^2$ (b) $x^2 + y^2 = a^2 + b^2$
(c) $4(x^2 - y^2) = a^2 - b^2$ (d) $x^2 - y^2 = a^2 - b^2$

19. The maximum number of points on the circle which have abscissa and ordinate both are numerically equal are -

(a) one (b) two
(c) four (d) three

20. A circle touches the y-axis at (0, 2) and has an intercept of 4 units on the positive side of the x-axis. Then the equation of the circle is -

(a) $x^2 + y^2 - 4(\sqrt{2}\,x + y) + 4 = 0$ (b) $x^2 + y^2 - 4(x + \sqrt{2}\,y) + 4 = 0$
(c) $x^2 + y^2 - 2(\sqrt{2}\,x + y) + 4 = 0$ (d) None of these

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SECTION - III

MULTIPLE CHOICE QUESTIONS

1. The points (2, 3), (0, 2), (4, 5) and (0, t) are concyclic if the value of t is
(a) 2 (b) 1
(c) 17 (d) 19

2. An equation of a line passing through the point (–2, 11) and touching the circle $x^2 + y^2 = 25$ is
(a) $4x + 3y = 25$ (b) $3x + 4y = 38$
(c) $24x – 7y + 125 = 0$ (d) $7x + 24y – 230 = 0$

3. If θ is the angle subtended at $P(x_1, y_1)$ by the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then
(a) $\cot\theta = \dfrac{\sqrt{S_1}}{\sqrt{g^2+f^2-c}}$ (b) $\cot\dfrac{\theta}{2} = \dfrac{\sqrt{S_1}}{\sqrt{g^2+f^2-c}}$
(c) $\tan\theta = \dfrac{2\sqrt{g^2+f^2-c}}{\sqrt{S_1}}$ (d) $\theta = 2\tan^{-1}\left(\dfrac{\sqrt{g^2+f^2-c}}{\sqrt{S_1}}\right)$

4. The centre of the circle(s) passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$, is (are)
(a) (3/2, 1/2) (b) (1/2, 3/2)
(c) $(1/2, 2^{1/2})$ (d) $(1/2, –2^{1/2})$

5. If two vertices of an equilateral triangle are (–1, 0) and (1, 0), equation of its circumcircle is
(a) $\sqrt{3}x^2 + \sqrt{3}y^2 + 2y - \sqrt{3} = 0$ (b) $2x^2 + 2y^2 + \sqrt{3}y - 2 = 0$
(c) $\sqrt{3}x^2 + \sqrt{3}y^2 - 2y - \sqrt{3} = 0$ (d) $3x^2 + 2y^2 - \sqrt{3}y - 2 = 0$

6. The equation of the tangents drawn from the origin to the circle $x^2 + y^2 – 2px – 2qy + q^2 = 0$ are perpendicular if
(a) p = q (b) p + q = 0
(c) $p^2 + q^2 = 1$ (d) q = 0

7. If $S \equiv x^2 + y^2 – 2x – 4y – 4 = 0$, $L \equiv 2x + 2y – 15 = 0$ and P(3, 4) represent a circle, a line and a point then
(a) L is a tangent to S at P (b) L is polar of P with respect to S
(c) L is the chord of contact of P with respect to S (d) P is inside S and L is outside S

8. A rectangle ABCD is inscribed in the circle $x^2 + y^2 + 3x + 12y + 2 = 0$. If the coordinates of A and B are respectively (3, –2) and (–2, 0), then the coordinates of
(a) C are (–6, –10) (b) C are (1, 12)
(c) D are (–1, –12) (d) D are (6, 10)

9. Equation of the circle that touches the circle $x^2 + y^2 = 1$ and passes through the point (4, 3) is
(a) $5(x^2 + y^2) – 24x – 18y + 25 = 0$ (b) $x^2 + y^2 – 24x – 18y – 25 = 0$
(c) $5(x^2 + y^2) – 16x – 12y + 25 = 0$ (d) $5(x^2 + y^2) – 16x – 12y – 25 = 0$

10. The circles $x^2 + y^2 – 8x + 2y + 8 = 0$ and $x^2 + y^2 – 2x – 6y + 10 – a^2 = 0$ have exactly two common tangents then a can be in
(a) $(-\infty, -2)$ (b) $(2, \infty)$
(c) (–8, –3] (d) [3, 8)

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SECTION - IV

COMPREHENSION TYPE QUESTIONS

Write up – I

Consider the circle $x^2 + y^2 = r^2$ and the point Q(a, b). There are infinite number of chords of the circle which subtend a right angle at Q. They can be imagined thus. Draw a ray QA and a perpendicular ray QB. Suppose they intersect the circle at A and B respectively. Then AB is a chord subtending a right angle at Q. Let P be the mid-point of AB.

Now, imagine that the two rays rotate in anti-clockwise direction, keeping the angle between them 90° always and complete one revolution. A and B are moving on the circle and we imagine that they are joined always and we concentrate our attention on their mid-point P which also moves and comes to the initial position as the chords complete one revolution. Thus, we can see all the chords subtending a right angle at Q.

The length of the chord of the circle $x^2 + y^2 = r^2$ having (x_1, y_1) as its mid-point is $2\sqrt{-S_{11}}$ where $S_{11} = x_1^2 + y_1^2 - r^2$. The locus of the point of intersection of perpendicular tangents to a circle is a concentric circle called the director circle of the given circle. The director circle of $x^2 + y^2 = r^2$ is $x^2 + y^2 = 2r^2$. In a right-angled triangle, the mid-point of the hypotenuse is the circumcentre and circumcentre is equidistant from the three vertices of the triangle.

1. The locus of the mid-points of the chords of the circle $x^2 + y^2 = r^2$ which subtend a right angle at the fixed point Q(a, b) is

(a) $x^2 + y^2 - ax - by + a^2 + b^2 - r^2 = 0$

(b) $x^2 + y^2 + ax + by - a^2 - b^2 + r^2 = 0$

(c) $2(x^2 + y^2) - 2(ax + by) + a^2 + b^2 - r^2 = 0$

(d) $2(x^2 + y^2) + 2(ax + by) - a^2 - b^2 + r^2 = 0$

2. If the locus of the previous question is a null set, then Q(a, b) must lie

(a) on the circle $x^2 + y^2 = r^2$

(b) on the director circle of $x^2 + y^2 = r^2$

(c) between the circle $x^2 + y^2 = r^2$ and its director circle

(d) outside the director circle of $x^2 + y^2 = r^2$

3. The locus of the point Q(a, b) such that the locus of first question consists of only one point is

(a) a circle (b) a pair of circles
(c) the region between two concentric circles (d) interior of a circle

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Write up – II

Let three circles

$S_1 \equiv x^2 + y^2 + 4y - 1 = 0$

$S_2 \equiv x^2 + y^2 + 6x + 4y + 8 = 0$

& $S_3 \equiv x^2 + y^2 - 4x - 4y - 37 = 0$

C_1, C_2 and C_3 are their centers respectively P is the point from which length of tangent to the three circles are equal. P_1, P_2 & P_3 are point of contact of tangent from P to the circles S_1, S_2 & S_3 respectively, then

4. Which of the following statement is true

(a) S_1 and S_2 touches each other internally

(b) Each of S_1, S_2 and S_3 intersect the other two

(c) Each of S_1, S_2 & S_3 touches the other two

(d) S_1 & S_3 touches each other internally

5. Co-ordinate of point P are

(a) (3, 3) (b) (3, –3)

(c) (– 3, 3) (d) $\left(-\frac{3}{2}, -\frac{15}{4}\right)$

6. Area of ΔPP_2P_3 is

(a) $\frac{3}{2}$ (b) $\frac{27}{2}$

(c) $\frac{9}{2}$ (d) none of these

MATCHING TYPE PROBLEM

1. (a) The circle $x^2 + y^2 - 6x - 10y + c = 0$ does not intersect or touch the co-ordinate axes and has (1, 4) as it's interior point. Exhaustive range of 'c' is -

   (b) The circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other. The co-ordinates of the corresponding point of contact is -

   (c) Co-ordinates of the mid point of the segment cut by the circle $x^2 + y^2 - 6x + 2y - 54 = 0$ on the line $2x - 5y + 18 = 0$ is -

   (d) Two circles 'C_2' and 'C_1' intersect in such a way that their common chord is of maximum length. Centre of C_1 is (1, 2) and its radius is 3 units. Radius of C_2 is 5 units. If slope of common chord is $\frac{3}{4}$ then centre of C_2 can be

   (P) $\left(-\frac{17}{5}, \frac{6}{5}\right)$

   (Q) (3, 1)

   (R) (2, 0)

   (S) (25, 29)

2. (a) The angle between a pair of tangents drawn from a point P to the circle $x^2 + y^2 + 4x - 6y + 9\sin^2 a + 13\cos^2 a = 0$ is 2a. The equation of the locus of the point P is

   (b) Equation of the circle through the origin and belonging to the co-axial system, of which the limiting points are (1, 2), (4, 3) is

   (c) The lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 square units. An equation of this circle is ($\pi = 22/7$)

   (d) The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length 3a is

   (P) $x^2 + y^2 = 4a^2$

   (Q) $x^2 + y^2 - 2x + 2y = 47$

   (R) $2x^2 + 2y^2 - x - 7y = 0$

   (S) $x^2 + y^2 + 4x - 6y + 9 = 0$

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ASSERTION-REASON TYPE QUESTIONS

In the following questions containing two statements viz. **Assertion (A)** & **Reason (R)**. To choose the correct answer.

Mark (a) if both A and R are correct & R is the correct explanation for A.

Mark (b) if both A & R are correct but R is not correct explanation for A

Mark (c) if A is true but R is false

Mark (d) if A is false but R is true

1. **A :** The triangle formed by common tangents to the circles $x^2 + y^2 - 2x = 0$, $x^2 + y^2 + 6x = 0$ is equilateral.

   R : If radii of circles are in 1 : 3 ratio, then common tangents will form an equilateral triangle

2. **A :** If the chord of contact of a circle of area 36π sq. units for a point P is 3 units away from the center of the circle O. The distance of the point P from the point O is 9 units

   R : The quadrilateral PAOB (A & B are the points of tangency for the tangents drawn from P) is a cyclic quadrilateral and $\sin 30^o = \frac{1}{2} = \cos 60^o$, also $\angle PAO = \frac{\pi}{2}$.

3. **A :** In a cyclic quadrilateral ABCD. The diagonals AC = BD = 4 & the diameter CD = 5. Then $AB = \frac{7}{5}$.

   R : In the cyclic quadrilateral ABCD, AC . BD = AB . CD + BC . AD

4. **A :** The tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + f^2 = 0$ are perpendicular, if f = g

   R : The director circle for $x^2 + y^2 + 2gx + 2fy + f^2 = 0$ is $x^2 + y^2 + 2gx + 2fy - g^2 = 0$

5. **A :** Line y = 2x + 2 cuts the circle $x^2 + y^2 = 1$ in two distinct points. The equation of circle with these points as end of its diameter is $5x^2 + 5y^2 + 8x - 4y + 3 = 0$.

   R : x_1 & x_2 are the roots of the equation $5x^2 + 8x + 3 = 0$, & y_1, y_2 are the roots of the equation $y^2 - \frac{4}{5}y = 0$.

6. **A :** Two circles neither touch nor intersect each other. Two circles have four common tangents.

   R : Sume of radii < distance between centres of two circle

   then centres of circles does not lie in each other.

7. **A :** Common chord of two circles is perpendicular to line joining then centres

   R : Radical axis of two circle is perpendicular to line joining then centre.

SECTION - V

PROBLEMS ASKED IN IIT-JEE

Subjective Problems

1. Circles with radii 3, 4 and 5 touch each other externally if P is the point of intersection of tangents to these circles at their points of contact. Find the distance of P from the points of contact. **[2005]**

2. A circle touches the line $2x + 3y + 1 = 0$ at the point $(1, -1)$ and is orthogonal to circle whose one pair of diametrically opposite end points are $(3, 0)$ and $(1, -3)$. Find the equation of circle. **[2004]**

3. Let T_1, T_2 be two tangents drawn from $(-2, 0)$ onto the circle $C : x^2 + y^2 = 1$. Determine the circles touching C and having T_1, T_2 as their pair of tangents. Further, find the equation of all possible common tangents to these circles, when taken two at a time. **[1999]**

4. C_1 and C_2 are two concentric circles, the radius of C_2 being twice that of C_1. From a point P on C_2, tangents PA and PB are drawn to C_1. Prove that the centroid of the triangle PAB lies on C_1. **[1998]**

5. The chords of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to the circle $x^2 + y^2 = 1$ pass through the point .... **[1997 Re-exam]**

6. Let C be any circle with centre $(0, \sqrt{2})$. Prove that at the most two rational points can be there on C. (A rational point is a point both of whose co-ordinates are rational numbers) **[1997 Re-exam]**

7. Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on the curve. A line drawn from the point P intersects the curve at point Q and R. If the product PQ. PR is independent of the slope of the line, then show that the curve is a circle **[1997 (Cancelled)]**

8. Two vertices of an equilateral triangle are $(-1, 0)$ and $(1, 0)$, and its third vertex lies above the x-axis. The equation of its circumcircle is .... **[1997 (Cancelled)]**

9. For each natural number k, let C_k denote the circle with radius k centimetres and centre at the origin. On the circle C_k, a particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n then n = ... **[1997 Re-exam]**

10. Find the intervals of values of a for which the line $y + x = 0$ bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2 + 2y^2 - (1 + \sqrt{2}\,a)x - (1 - \sqrt{2}\,a)\,y = 0$. **[1996]**

11. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle with AB as a diameter is .... **[1996]**

12. The circle $x^2 + y^2 = 1$ cuts the x-axis at P and another circle with centre at Q and variable radius intersects the first circle at R above the axis and the line segment PQ at S. Find the maximum area of the triangle QSR. **[1994, 5]**

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13. The equation of the locus of the mid points of the chords of the circle $4x^2 + 4y^2 - 12x + 4y + 1 = 0$ that subtend an angle of $\frac{2\pi}{3}$ at its centre is .... **[1993]**

14. Three circles touch one another externally. The tangent at their points of contact meet at a point whose distance from a point of contact is 4. Find the ratio of the product of the radii to the sum of the radii of the circles. **[1992, 4]**

15. Let a circle be given by $2x(x - a) + y(2y - b) = 0$, $(a \neq 0, b \neq 0)$. Find the condition on a and b if two chords, each-bisects by the x-axis, can be drawn to the circle from $(a, b/2)$. **[1992, 6]**

16. If a circle passes through the points of intersection of the coordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ is .... **[1991]**

17. Two circles, each of radius 5 units, touch each other at (1, 2). If the equation of their common tangent is $4x + 3y = 10$, find equations of the circles. **[1991]**

18. A circle touches the line $y = x$ at a point P such that $OP = 4\sqrt{2}$, where O is the origin. The circle contains the point (–10, 2) in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$. Determine the equation of the circle. **[1990, 95]**

Objective Problems

19. Let ABCD be a quadrilateral with area 18, with side AB parallel to the side CD and AB = 2CD. Let AD be perpendicular to AB and CD. If a circle is drawn insicde the quadrilateral ABCD touching all the sides, then its radius is

(a) 3 (b) 2

(c) 3/2 (d) 1 **[2007]**

20. The locus of centre of the circle which touches the circle $x^2 + (y - 1)^2 = 1$ externally and also touches x-axis is

(a) $\{(x, y) : x^2 + (y - 1)^2 = 4\} \cup \{(x, y) : y < 0\}$ (b) $\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y < 0\}$

(c) $\{(x, y) : x^2 = y\} \cup \{(0, y) : y < 0\}$ (d) $\{(x, y) : x^2 = 4y\} \cup \{(x, y) : y > 0\}$ **[2005]**

21. The area of the equilateral triangle which contains three coins of unity radius is

(a) $6 + 4\sqrt{3}$ sq. units (b) $8 + \sqrt{3}$ sq. units

(c) $4 + \frac{7\sqrt{3}}{2}$ sq. units (d) $12 + 2\sqrt{3}$ sq. units **[2005]**

22. If one of the diameters of circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord with centre (2,1), then the radius of circle is- **[2004]**

(a) $\sqrt{3}$ (b) $\sqrt{2}$

(c) 3 (d) 2

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23. The centre of circle inscribed in square formed by the lines $x^2 - 8x + 12 = 0$ and $y^2 - 14y + 45 = 0$, is- **[2003]**

(a) (4, 9) (b) (7, 4)

(c) (9, 4) (d) (4, 7)

24. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then the locus of the centroid of the triangle PAB as P moves on the circle is – **[2001]**

(a) a parabola (b) a circle

(c) an ellipse (d) a pair of straight lines

25. The triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have co-ordinates (3, 4) and (–4, 3) respectively, then $\angle QPR$ is equal is - **[2000]**

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

26. If the circles $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is **[2000]**

(a) $2 \text{ or } -\frac{3}{2}$ (b) $-2 \text{ or } -\frac{3}{2}$

(c) $2 \text{ or } \frac{3}{2}$ (d) $-2 \text{ or } \frac{3}{2}$

27. The number of common tangents to the circle $x^2 + y^2 = 4$ and $x^2 + y^2 - 6 - 8y = 24$ is - **[1998]**

(a) 0 (b) 1

(c) 3 (d) 4

28. The angle between a pair of tangents drawn from a point P to the circle

$$x^2 + y^2 + 4x - 6y + 9 \sin^2 a + 13 \cos^2 a = 0 \text{ is } 2a.$$

The equation of the locus of the point P is **[1996]**

(a) $x^2 + y^2 + 4x - 6y + 4 = 0$ (b) $x^2 + y^2 + 4x - 6y - 9 = 0$

(c) $x^2 + y^2 + 4x - 6y - 4 = 0$ (d) $x^2 + y^2 + 4x - 6y + 9 = 0$

29. The locus of the centre of a circle which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the y-axis is given by the equation - **[1993, 96]**

(a) $x^2 - 6x - 10y + 14 = 0$ (b) $x^2 - 10x - 6y + 14 = 0$

(c) $y^2 - 6x - 10y + 14 = 0$ (d) $y^2 - 10x - 6y + 14 = 0$

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30. The centre of a circle passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is -

(a) (3/ 2, 1/2) (b) (1/2, 3/2)

(c) (1/2, 1/2) (d) $(1/2, -2^{1/2})$ **[1992]**

COMPREHENSION TYPE QUESTIONS

31. **Write-up :** **[2006]**

Let ABCD be a square of side length 2 units. C_2 is the circle through vertices A, B, C D and C_1 is the circle touching all the sides of the square ABCD. L is the line through A

(i) If P is a point on C_1 and Q is a point on C_2, then $\frac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$ is equal to

(a) 0.75 (b) 1.25

(c) 1 (d) 0.5

(ii) A circle touches the line L and the circle C1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is

(a) ellipse (b) hyperbola

(c) parabola (d) pair of straight lines

(iii) A line M through A is drawn parallel to BD. Points S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1, then area of $\Delta T_1 T_2 T_3$ is

(a) 1/2 sq. units (b) 2/3 sq. units

(c) 1 sq. units (d) 2 sq. units

MATCHING TYPE PROBLEM

32. Match the statements in on Column I with statements in Column II. **[2007]**

Column I	**Column II**
(a) Two intersecting circles	(P) have a common tangent
(b) Two mutually external circles	(Q) have a common normal
(c) Two circles, one strictly inside the other	(R) do not have a common tangent
	(S) do not have a common normal

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ASSERTION-REASON TYPE QUESTIONS

33. Tangents are drawn from the point (17, 7) to the circle $x^2 + y^2 = 169$. **[2007]**

Statement – 1: The tangents are mutually perpendicular.

because

Statement – 2: The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$

(a) Statement–1 is True, Statement–2 is True; Statement-2 is a correct explanation for Statement-1

(b) Statement–1 is True, Statement–2 is True; Statement–2 NOT a correct explanation for Statement–1

(c) Statement–1 is True, Statement–2 is False

(d) Statement–1 is False, Statement–2 is True

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ANSWERS

EXERCISE - 1

1. (i) $x^2 + y^2 = 25$ (ii) $x^2 + y^2 = 289$
   (iii) $x^2 + y^2 = 100$ (iv) $x^2 + y^2 = 144$
   (v) $x^2 + y^2 = 2(a^2 + b^2)$
2. (i) $x^2 + (y - 2)^2 = 4$ (ii) $(x - 8)^2 + (4 - 15)^2 = 144$
   (iii) $(x + 3)^2 + (y + 2)^2 = 16$ (iv) $(x - 3)^2 + (y - 4)^2 = 36$
   (v) $[x - (a + b)]^2 + [y -(a - b)]^2 = 2(a^2 + b^2)$
3. (i) $(-2, 3), 4$ (ii) $(5, -6), 9$
   (iii) $(-4, -1), 3$ (iv) $\left(-1, \sqrt{\frac{3}{2}}\right), \sqrt{\frac{35}{12}}$
   (v) $(2a, -5b), 36$
4. (i) $(x - 3)^2 + (y - 5)^2 = 13$ (ii) $(x + 1)^2 + (y - 4)^2 = 45$
   (iii) $(x - y)^2 + (y + 9)^2 = 97$(iv) $(x + 2)^2 + (y + 6)^2 = 1$
   (v) $(x - a)^2 + (y + b)2 = {}^2 (a^2 + b)^2$
5. 11, 1

EXERCISE - 2

1. $(x - 4) (x - 10) + (y - 8) (y - 2) = 0$

EXERCISE - 3

2. $(x - 6)^2 + (y - 3)^2 = 25, (x - 6)^2 + (y + 3)^2 = 25$
3. $(x - 6)^2 + (y + 9)^2 = 85, - 16$
4. $(x + 1)^2 + (y - 2)^2 = 5$

EXERCISE - 4

3. $(x - 2)^2 + (y - 10)^2 = 100$
4. $(x - 5)^2 + (y + 3)^2 = 25$
5. $(x + 1)^2 + (y - 3)^2 = 9, (x - 11)^2 + (y - 15)^2 = 225$
6. (7, – 1) & 3, (19, 11) & 15
7. $(x - a)^2 + (y - 3a + 2)^2 = (3a - 2)^2,\ (x - a)^2 + \left(y+\frac{3a-2}{9}\right)^2 = \left(\frac{3a-2}{9}\right)^2$
8. $\left(\frac{15}{8}, \frac{5}{2}\right), \left(\frac{3}{2}, 4\right)$
9. $(x \pm 3)^2 + \left(y \pm 3\sqrt{2}\right)^2 = 18$
10. $[x-(5+10\pi)]^2 + (y-5)^2 = 25$

EXERCISE - 5

4. 2

EXERCISE - 6

1. $4x + 3y = 25$
2. $2x + 5y = 0$
5. (9, 1)
8. $\left(x-\frac{3}{2}\right)^2 + \left(y-\frac{5}{2}\right)^2 = \frac{1}{2}$

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EXERCISE - 7

1. y = 4, x = 5, (3, 4), (5, 2)

EXERCISE - 8

1. (i) $\sqrt{20}$ (ii) $\sqrt{24}$

EXERCISE - 9

1. (i) 5x + 3y = 14 (ii) 3x – y – 7 = 0

EXERCISE - 10

1. (i) $x^2+y^2=9\left(\frac{2x+3y}{6}\right)^2$ (ii) $x^2+y^2+2(x-y)\left(\frac{y-2x}{4}\right)=8\left(\frac{y-2x}{4}\right)^2$

EXERCISE - 11

1. $4[(x-2)^2+(y+3)^2-5]-3[(x+4)^2+(y-2)^2-34]=0$
2. $7(x^2+y^2-2x-7y+7)+5(x^2+y^2-6x-5y+9)=0$

EXERCISE - 12

2. $(x+1)^2+\left(y+\frac{3}{2}\right)^2=\frac{49}{4}$

EXERCISE - 13

1. $\left(\frac{10}{7}\right)$
3. x = 1, (1, 3) & (1, –3)
4. 2x + 3y = 7

EXERCISE - 14

1. x = y, (1, 1) & (–2, –2)

EXERCISE - 15

1. (i) x – y = 2, (1, –1) (ii) 4x + 3y = 21, $\left(\frac{6}{5}, \frac{27}{5}\right)$

EXERCISE - 16

1. ax + by = 0
2. $x^2+y^2=68$
3. 3y – 2x = 2

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SECTION - I

Subjective Questions

LEVEL – I

2. (2, 8), (6, 4)

4. (i) $\left(\frac{a+b}{2}, \frac{c+d}{2}\right), \frac{\sqrt{(a-b)^2+(c-d)^2}}{2}$ (ii) (0, – a), 2a

6. $x^2 + y^2 - 10x - 6y + 9 = 0$ 8. $x - y + 10 = 0$

9. $x^2 + y^2 - 6x + 4y = 0, x^2 + y^2 + 2x - 8y + 4 = 0$

LEVEL – II

4. $(a - b)^2 = 2c^2$ 6. $y = 2, 4x - 3y = 10$ 9. $(y - x)^2 = 0$

LEVEL – III

3. (i) $\left(1, \frac{4}{3}\right)$, (ii) $\left(\frac{17}{3}, 6\right)$

SECTION - II

Single Choice Questions

1.	(b)	2.	(c)	3.	(c)	4.	(d)
5.	(a)	6.	(b)	7.	(a)	8.	(c)
9.	(d)	10.	(a)	11.	(b)	12.	(b)
13.	(d)	14.	(b)	15.	(c)	16.	(c)
17.	(b)	18.	(c)	19.	(c)	20.	(a)

SECTION - III

Multiple Choice Questions

1.	(a), (c)	2.	(a), (c)	3.	(b), (d)	4.	(c), (d)
5.	(a), (c)	6.	(a), (b)	7.	(b), (d)	8.	(a), (c)
9.	(a), (d)	10.	(a), (b), (c), (d)				

SECTION - IV

Comprehension Type Questions

1.	(c)	2.	(d)	3.	(b)	4.	(d)
5.	(d)	6.	(a)				

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Matching Type Questions

1. (a) – (S), (b) – (R), (c) – (Q), (d) – (P)
2. (a) – (S), (b) – (R), (c) – (Q), (d) – (P)

Assertion Reason Type Questions

1.	(a)	2.	(d)	3.	(a)	4.	(c)
5.	(a)	6.	(a)	7.	(a)		

SECTION - V

Question Asked in IIT-JEE

1. $\sqrt{5}$
2. $x^2 + y^2 + 6x + 14y + 6 = 0$
3. $x^2 + y^2 - 1 + 2(x - \sqrt{3}y + 2) = 0$; $x^2 + y^2 - 1 + 4(x + \sqrt{3}y + 2) = 0$; $x + 3\sqrt{3}y + 2 = 0$
5. $\left(\frac{1}{2}, \frac{1}{4}\right)$
8. $x^2 + y^2 - \frac{2y}{\sqrt{3}} - 1 = 0$
9. 7
10. $a \in (-\infty, -2) \cup (2, \infty)$
11. $x^2 + y^2 - x - y = 0$
12. $\sqrt{\frac{4}{3\sqrt{3}}}$ Sq. unit
13. $16(x^2 + y^2) - 48x + 16y + 31 = 0$
14. 16
15. $a^2 > b^2$
16. 2
17. $x^2 + y^2 - 10x - 10y + 25 = 0$ and $x^2 + y^2 + 6x + 2y - 15 = 0$
18. $x^2 + y^2 + 18x - 2y + 32 = 0$

20.	(b)	21.	(a)	22.	(c)	23.	(d)
24.	(b)	25.	(c)	26.	(c)	27.	(b)
28.	(d)	29.	(d)	30.	(d)		

31. (i) – (a) (ii) – (c) (iii) – (c)
32. (a) – (P), (Q), (b) – (P), (Q), (c) – (Q), (R)
33. (a)

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