Senior School Certificate Examination-2020

Marking Scheme - Mathematics 65/1/1

***General instructions*:-**

1. You are aware that evaluation is the most important process in the actual and correct assessment of the candidates. A small mistake in evaluation may lead to serious problems which may affect the future of the candidates, education system and teaching profession. To avoid mistakes, it is requested that before starting evaluation, you must read and understand the spot evaluation guidelines carefully. **Evaluation is a 10-12 days mission for all of us. Hence, it is necessary that you put in your best efforts in this process.**

2. Evaluation is to be done as per instructions provided in the Marking Scheme. It should not be done according to one's own interpretation or any other consideration. Marking Scheme should be strictly adhered to and religiously followed. **However, while evaluating, answers which are based on latest information or knowledge and/or are innovative, they may be assessed for their correctness otherwise and marks be awarded to them.**

3. The Head-Examiner must go through the first five answer books evaluated by each evaluator on the first day, to ensure that evaluation has been carried out as per the instructions given in the Marking Scheme. The remaining answer books meant for evaluation shall be given only after ensuring that there is no significant variation in the marking of individual evaluators.

4. Evaluators will mark(√) wherever answer is correct. For wrong answer 'X"be marked. Evaluators will not put right kind of mark while evaluating which gives an impression that answer is correct and no marks are awarded. This is most common mistake which evaluators are committing.

5. If a question has parts, please award marks on the right-hand side for each part. Marks awarded for different parts of the question should then be totaled up and written in the left-hand margin and encircled. This may be followed strictly.

6. If a question does not have any parts, marks must be awarded in the left-hand margin and encircled. This may also be followed strictly.

7. If a student has attempted an extra question, answer of the question deserving more marks should be retained and the other answer scored out.

8. No marks to be deducted for the cumulative effect of an error. It should be penalized only once.

9. A full scale of marks __________(example0-100 marks as given in Question Paper) has to be used. Please do not hesitate to award full marks if the answer deserves it.

10. Every examiner has to necessarily do evaluation work for full working hours i.e. 8 hours every day and evaluate 20 answer books per day in main subjects and 25 answer books per day in other subjects (Details are given in Spot Guidelines).

11. Ensure that you do not make the following common types of errors committed by the Examiner in the past:-

- Leaving answer or part thereof unassessed in an answer book.
- Giving more marks for an answer than assigned to it.
- Wrong totaling of marks awarded on a reply
- Wrong transfer of marks from the inside pages of the answer book to the title page.
- Wrong question wise totaling on the title page.
- Wrong totaling of marks of the two columns on the title page.
- Wrong grand total.
- Marks in words and figures not tallying.
- Wrong transfer of marks from the answer book to online award list.
- Answers marked as correct, but marks not awarded. (Ensure that the right tick mark is correctly and clearly indicated. It should merely be a line. Same is with the X for incorrect answer.)
- Half or a part of answer marked correct and the rest as wrong, but no marks awarded.

12. While evaluating the answer books if the answer is found to be totally incorrect, it should be marked as cross (X) and awarded zero (0)Marks.

13. Any unassessed portion, non-carrying over of marks to the title page, or totaling error detected by the candidate shall damage the prestige of all the personnel engaged in the evaluation work as also of the Board. Hence, in order to uphold the prestige of all concerned, it is again reiterated that the instructions be followed meticulously and judiciously.

14. The Examiners should acquaint themselves with the guidelines given in the Guidelines for spot Evaluation before starting the actual evaluation.

15. Every Examiner shall also ensure that all the answers are evaluated, marks carried over to the title page, correctly totaled and written in figures and words.

16. The Board permits candidates to obtain photocopy of the Answer Book on request in an RTI application and also separately as a part of the re-evaluation process on payment of the processing charges.

QUESTION PAPER CODE 65/1/1

EXPECTED ANSWER/VALUE POINTS

SECTION – A

Question Numbers 1 to 20 carry 1 mark each.

Question Numbers 1 to 10 are multiple choice type questions. Select the correct option.

Q.No.		Marks
1.	If A is a square matrix of order 3 and $|A| = 5$, then the value of $|2A'|$ is **(A)** –10 **(B)** 10 **(C)** –40 **(D)** 40 $\boxed{\textbf{Ans: } \text{(D) } 40}$	1
2.	If A is a square matrix such that $A^2 = A$, then $(I - A)^3 + A$ is equal to **(A)** I **(B)** 0 **(C)** $I - A$ **(D)** $I + A$ $\boxed{\textbf{Ans: } \text{(A) } I}$	1
3.	The principal value of $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)$ **(A)** $\frac{2\pi}{5}$ **(B)** $\frac{-2\pi}{5}$ **(C)** $\frac{3\pi}{5}$ **(D)** $\frac{-3\pi}{5}$ $\boxed{\textbf{Ans: } \text{(B) } -\frac{2\pi}{5}}$	1
4.	If the projection of $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ on $\vec{b} = 2\hat{i} + \lambda\hat{k}$, is zero, then the value of λ is **(A)** 0 **(B)** 1 **(C)** $\frac{-2}{3}$ **(D)** $\frac{-3}{2}$ $\boxed{\textbf{Ans: } \text{(C) } -\frac{2}{3}}$	1
5.	The vector equation of the line passing through the point (–1, 5, 4) and perpendicular to the plane z = 0 is **(A)** $\vec{r} = -\hat{i} + 5\hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j})$ **(B)** $\vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}$ **(C)** $\vec{r} = \hat{i} - 5\hat{j} - 4\hat{k} + \lambda\hat{k}$ **(D)** $\vec{r} = \lambda\hat{k}$ $\boxed{\textbf{Ans: } \text{(B) } \vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}}$	1
6.	The number of arbitrary constants in the particular solution of a differential equation of second order is (are) **(A)** 0 **(B)** 1 **(C)** 2 **(D)** 3 $\boxed{\textbf{Ans: } \text{(A) } 0}$	1

7. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 x dx$

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2

Ans: (D) 2 **1**

8. The length of the perpendicular drawn from the point (4, –7, 3) on the y-axis is

(A) 3 units **(B)** 4 units **(C)** 5 units **(D)** 7 units

Ans: (C) 5 units **1**

9. If A and B are two independent events with $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, then $P(B' | A)$ is equal to

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3}{4}$ **(D)** 1

Ans: (C) $\frac{3}{4}$ **1**

10. The corner points of the feasible region determined by the system of linear inequalities are (0, 0), (4,0), (2, 4) and (0, 5). If the maximum value of $z = ax + by$, where a, b > 0 occurs at both (2, 4) and (4,0), then

(A) a = 2b **(B)** 2a = b **(C)** a = b **(D)** 3a = b

Ans: (A) $a = 2b$ **1**

Fill in the blanks in questions numbers 11 to 15

11. A relation R in a set A is called ______________, if $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

Ans: Symmetric **1**

12. The greatest integer function defined by $f(x) = [x], 0 < x < 2$ is not differentiable at x = ___________.

Ans: 1

13. If A is a matrix of order 3 × 2, then the order of the matrix A′ is _________.

Ans: 2 × 3 **1**

OR

A square matrix A is said to be skew-symmetric, if ________

Ans: $A = -A' \left(\text{or, } A' = -A\right)$ **1**

14. The equation of the normal to the curve $y^2 = 8x$ at the origin is __________

Ans: $y = 0$ **1**

OR

The radius of a circle is increasing at the uniform rate of 3 cm/s. At the instant when the radius of the circle is 2 cm, its area increases at the rate of ____________ cm^2/s.

Ans: 12π **1**

15. The position vectors of two points A and B are $\overrightarrow{OA} = 2\hat{i} - \hat{j} - \hat{k}$ and $\overrightarrow{OB} = 2\hat{i} - \hat{j} + 2\hat{k}$, respectively. The position vector of a point P which divides the line segment joining A and B in the ratio 2 : 1 is _______

Ans: $2\hat{i} - \hat{j} + \hat{k}$ **1**

Question numbers 16 to 20 are very short answer type questions

16. If $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 3 \\ 3 & 3 & 5 \end{bmatrix}$, then find A (adj A).

Ans: $A \cdot \text{adj}(A) = |A|\ I$ **1/2**

$$\therefore\ A \cdot \text{adj}\ (A) = 2I \ \text{ or } \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ **1/2**

17. Find $\int x^4 \log x dx$

Ans: $\int x^4 \cdot \log x\ dx = \log x \cdot \frac{x^5}{5} - \int \frac{1}{x} \cdot \frac{x^5}{5} dx$ **1/2**

$$= \frac{x^5 \cdot \log x}{5} - \frac{x^5}{25} + c$$ **1/2**

OR

Find $\int \frac{2x}{\sqrt[3]{x^2+1}} dx$

Ans: Let, $x^2 + 1 = t \quad \therefore \quad 2x\ dx = dt$ **1/2**

$$\int \frac{2x}{\sqrt[3]{x^2+1}} dx = \int \frac{1}{\sqrt[3]{t}} dt = \int t^{-1/3} dt = \frac{3}{2} t^{2/3} + c$$

$$= \frac{3}{2}\left(x^2+1\right)^{2/3} + c$$ **1/2**

18. Evaluate $\int_1^3 |2x-1|\,dx$.

Ans: $\int_1^3 12x-11dx = \int_1^3 (2x-1)\,dx = \left[\frac{1}{4}(2x-1)^2\right]_1^3$ **1/2**

$= 6$ **1/2**

19. Two cards are drawn at random and one-by-one without replacement from a well-shuffled pack of 52 playing cards. Find the probability that one card is red and the other is black.

Ans: $\frac{{}^{26}C_1 \times {}^{26}C_1}{{}^{52}C_2} = \frac{26}{51}$ **1/2+1/2**

20. Find $\int \frac{dx}{\sqrt{9-4x^2}}$.

Ans: $\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{\sqrt{3^2-(2x)^2}}$ **1/2**

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + c$ **1/2**

SECTION-B

Question numbers 21 to 26 carry 2 marks each.

21. Prove that $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x, \frac{1}{\sqrt{2}} \le x \le 1$

Ans: Put $x = \cos\theta \Leftrightarrow \theta = \cos^{-1}x$ **1/2**

L.H.S. $= \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$= \sin^{-1}(2\cos\theta\ \sin\theta) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\cos^{-1}x = \text{R.H.S.}$ **$1\frac{1}{2}$**

OR

Consider a bijective function $f: R_+ \to (7, \infty)$ given by $f(x) = 16x^2 + 24x + 7$, where R_+ is the set of all positive real numbers. Find the inverse function of f.

Ans: Let $y = f(x) = 16x^2 + 24x + 7 = (4x+3)^2 - 2$ **1**

$\Rightarrow\ f^{-1}(y) = x = \frac{\sqrt{y+2}-3}{4}$ **1**

22. If $x = at^2, y = 2at$, then find $\frac{d^2y}{dx^2}$.

Ans: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$ **1**

$\frac{d^2y}{dx^2} = -\frac{1}{t^2} \cdot \frac{dt}{dx} = -\frac{1}{t^2} \cdot \frac{1}{2at} = -\frac{1}{2a\,t^3}$ **1**

23. Find the points on the curve $y = x^3 - 3x^2 - 4x$ at which the tangent lines are parallel to the line $4x + y - 3 = 0$.

Ans: $\frac{dy}{dx} = -4 \Rightarrow 3x^2 - 6x - 4 = -4$ **1**

$\Rightarrow 3x(x-2) = 0 \;\therefore\; x = 0\,;\, x = 2$ **1/2**

Points on the curve are (0, 0), (2, –12) **1/2**

24. Find a unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$ where $\vec{a} = 5\hat{i} + 6\hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 6\hat{j} + 2\hat{k}$.

Ans: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 6 & -2 \\ 7 & 6 & 2 \end{vmatrix} = 24\hat{i} - 24\hat{j} - 12\hat{k}$ **1**

Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$ **1**

OR

Find the volume of the parallelopiped whose adjacent edges are represented by $2\vec{a}, -\vec{b}$ and $3\vec{c}$, where $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 3\hat{k}$

Ans: Volume of the parallelopiped = $\begin{vmatrix} 2 & -2 & 4 \\ -3 & -4 & 5 \\ 6 & -3 & 9 \end{vmatrix}$ **1**

$= |-24| = 24$ **1**

25. Find the value of k so that the lines x = –y = kz and x – 2 = 2y + 1 = –z + 1 are perpendicular to each other.

Ans: The lines, $\frac{x}{1} = \frac{y}{-1} = \frac{z}{\frac{1}{k}}$ and $\frac{x-2}{1} = \frac{y+\frac{1}{2}}{\frac{1}{2}} = \frac{z-1}{-1}$ **1**

are perpendicular $\therefore\; 1 - \frac{1}{2} - \frac{1}{k} = 0 \Rightarrow k = 2$ **1**

26. The probability of finding a green signal on a busy crossing X is 30%. What is the probability of finding a green signal on X on two consecutive days out of three?

Ans: Probability of green signal on crossing $X = \frac{30}{100} = \frac{3}{10}$

Probability of not a green signal on crossing $X = 1 - \frac{3}{10} = \frac{7}{10}$ **1**

Probability of a green signal on X on two concecutative days out of three

$= \frac{3}{10} \times \frac{3}{10} \times \frac{7}{10} + \frac{7}{10} \times \frac{3}{10} \times \frac{3}{10} = \frac{63}{500}$ **1**

SECTION-C

Question numbers 27 to 32 carry 4 marks each.

27. Let N be the set of natural numbers and R be the relation on N × N defined by (a, b) R (c, d) iff ad = bc for all a, b, c, d ∈ N. Show that R is an equivalence relation.

Ans: Reflexive: For any (a, b) ∈ N × N

$a \cdot b = b \cdot a$

∴ (a, b) R (a, b) thus R is reflexive **1**

Symmetric: For (a, b), (c, d) ∈ N × N

(a, b) R (c, d) $\Rightarrow a \cdot d = b \cdot c$

$\Rightarrow c \cdot b = d \cdot a$

⇒ (c, d) R (a, b) ∴ R is symmetric $\mathbf{1\frac{1}{2}}$

Transitive : For any (a, b), (c, d), (e, f), ∈ N × N

(a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow a \cdot d = b \cdot c$ and $c \cdot f = d \cdot e$

$\Rightarrow a \cdot d \cdot c \cdot f = b \cdot c \cdot d \cdot e \Rightarrow a \cdot f = b \cdot e$

∴ (a, b) R (e, f), ∴ R is transitive $\mathbf{1\frac{1}{2}}$

∴ R is an equivalance Relation

28. If $y = e^{x^2 \cos x} + (\cos x)^x$, then find $\frac{dy}{dx}$.

Ans. Let $u = (\cos x)^x \Rightarrow y = e^{x^2 \cdot \cos x} + u$

$\therefore \frac{dy}{dx} = e^{x^2 \cdot \cos x}(2x \cdot \cos x - x^2 \cdot \sin x) + \frac{du}{dx}$ $\mathbf{1\frac{1}{2}}$

$\log u = \log (\cos x)^x \Rightarrow \log u = x . \log(\cos x)$

Differentiate w.r.t. "x"

$$\frac{1}{u}\frac{du}{dx} = \log(\cos x) - x\tan x \Rightarrow \frac{du}{dx} = (\cos x)^x\{\log(\cos x) - x\tan x\}$$ **2**

Therefore,

$$\frac{dy}{dx} = e^{x^2\cdot\cos x}(2x\cdot\cos x - x^2\cdot\sin x) + (\cos x)^x\{\log(\cos x) - x\tan x\}$$ **1/2**

29. Find $\int \sec^3 x dx$.

Ans. $\int \sec^3 x dx = \int \sec x\cdot\sec^2 x dx = \int\sqrt{1+\tan^2 x}\cdot\sec^2 x dx$ **$1\frac{1}{2}$**

(Put $\tan x = t$; $\sec^2 x\, dx = dt$) **1/2**

$$= \int\sqrt{1+t^2}\, dt$$

$$= \frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\log\left|t+\sqrt{1+t^2}\right| + c$$ **$1\frac{1}{2}$**

$$= \frac{\sec x\cdot\tan x}{2} + \frac{1}{2}\log|\tan x + \sec x| + c$$ **1/2**

30. Find the general solution of the differential equation $ye^y dx = (y^3 + 2xe^y)dy$.

Ans. $y\cdot e^y dx = \left(y^3 + 2xe^y\right)dy \Rightarrow y\cdot e^y\frac{dy}{dx} = y^3 + 2xe^y$

$$\therefore \frac{dx}{dy} - \frac{2}{y}x = y^2\cdot e^{-y}$$ **1**

I.F. (Integrating factor) $= e^{-2\int\frac{1}{y}dy} = e^{-2\log y} = e^{\log\frac{1}{y^2}} = \frac{1}{y^2}$ **1**

$\therefore$ Solution is

$$x\cdot\frac{1}{y^2} = \int y^2\cdot e^{-y}\cdot\frac{1}{y^2}dy + c = \int e^{-y}dy + c$$ **1**

$$\Rightarrow \frac{x}{y^2} = -e^{-y} + c \quad \text{or} \quad x = -y^2e^{-y} + cy^2$$ **1**

OR

Find the particular solution of the differential equation

$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$, given that $y = \frac{\pi}{4}$ at $x = 1$.

Ans. The differential equation can be written as:

$$\frac{dy}{dx} = \frac{y}{x} - \tan\frac{y}{x}, \text{ let } y = vx \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$ **1**

$$\Rightarrow \; v + x\frac{dv}{dx} = v - \tan v \;\Rightarrow\; \cot v \, dv = -\frac{1}{x}dx$$

Integrate both sides

$$\log \sin v = -\log|x| + \log c \Rightarrow \log \sin\frac{y}{x} = \log \frac{c}{x}$$ **2**

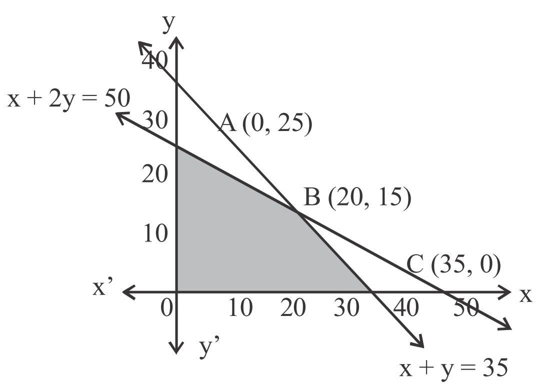
$$\Rightarrow \; x \cdot \sin\frac{y}{x} = c, \text{ Put } y = \frac{\pi}{4} \text{ and } x = 1$$

$$\Rightarrow \; \sin\frac{\pi}{4} = c \text{ or } c = \frac{1}{\sqrt{2}}$$ **1/2**

$\therefore$ Particular solution is $x \cdot \sin\left(\frac{y}{x}\right) = \frac{1}{\sqrt{2}}$ **1/2**

31. A furniture trader deals in only two items – chairs and tables. He has ₹ 50,000 to invest and a space to store at most 35 items. A chair costs him ₹ 1000 and a table costs him ₹ 2000. The trader earns a profit of ₹ 150 and ₹ 250 on a chair and table, respectively. Formulate the above problem as an LPP to maximise the profit and solve it graphically.

Ans.



Let No. of chairs = x, No. of tables = y

Then L.P. P. is:

Maximize (Profit) : Z = 150x + 250y **1**

Subject to : $x + y \leq 35$

$1000x + 2000y \leq 50000 \Rightarrow x + 2y \leq 50$ **1**

$x, y \geq 0$

Correct graph $1\frac{1}{2}$

Corner:	Value of Z
A(0, 25)	₹ 6250
B(20, 15)	₹ 6750 (Max)
C(35, 0)	₹ 5250

$\therefore$ Max (z) = ₹ 6750

Number of chairs = 20, Tables = 15 **1/2**

32. There are two bags, I and II. Bag I contains 3 red and 5 black balls and Bag II contains 4 red and 3 black balls. One ball is transferred randomly from Bag I to Bag II and then a ball is drawn randomly from Bag II. If the ball so drawn is found to be black in colour, then find the probability that the transferred ball is also black.

Ans. E_1 = Event that the ball transfered from Bag I is Black

E_2 = Event that the ball transfered from Bag I is Red

A = Event that the ball drawn from Bag II is Black **1/2**

$$P(E_1) = \frac{5}{8}; P(E_2) = \frac{3}{8}; P\left(\frac{A}{E_1}\right) = \frac{4}{8} = \frac{1}{2}; P\left(\frac{A}{E_2}\right) = \frac{3}{8}$$ **2**

Required Probability:

$$P\left(\frac{E_1}{A}\right)=\frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right)+P(E_2)\cdot P\left(\frac{A}{E_2}\right)}=\frac{\frac{5}{8}\cdot\frac{1}{2}}{\frac{5}{8}\cdot\frac{1}{2}+\frac{3}{8}\cdot\frac{3}{8}}=\frac{20}{29}$$ $1\frac{1}{2}$

OR

An urn contains 5 red, 2 white and 3 black balls. Three balls are drawn, one-by-one, at random without replacement. Find the probability distribution of the number of white balls. Also, find the mean and the variance of the number of white balls drawn.

Ans. Let X = No. of white balls = 0, 1, 2

X :	0	1	2	1/2
P(X) :	$\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}=\frac{7}{15}$	$3\times\frac{8}{10}\times\frac{7}{9}\times\frac{2}{8}=\frac{7}{15}$	$3\times\frac{2}{10}\times\frac{1}{9}\times\frac{8}{8}=\frac{1}{15}$	$1\frac{1}{2}$
X · P(X) :	0	$\frac{7}{15}$	$\frac{2}{15}$	1/2
$X^2P(X)$:	0	$\frac{7}{15}$	$\frac{4}{15}$	

$$\text{Mean} = \sum XP(X)=\frac{9}{15}=\frac{3}{5}$$ 1/2

$$\text{Variance} = \sum X^2P(x)-\left[\sum X\,P(X)^2\right]=\frac{11}{15}-\left[\frac{3}{5}\right]^2=\frac{28}{75}$$ 1

SECTION-D

Question numbers 33 to 36 carry 6 marks each.

33. If $A=\begin{bmatrix}1 & 2 & -3\\ 3 & 2 & -2\\ 2 & -1 & 1\end{bmatrix}$, then find A^{-1} and use it to solve the following system of the equations:

$x+2y-3z=6$

$3x+2y-2z=3$

$2x-y+z=2$

Ans. $|A|=7\,;\ \text{adj}(A)=\begin{bmatrix}0 & 1 & 2\\ -7 & 7 & -7\\ -7 & 5 & -4\end{bmatrix}\ ;\ A^{-1}=\frac{1}{|A|}\text{adj}\,A=\frac{1}{7}\begin{bmatrix}0 & 1 & 2\\ -7 & 7 & -7\\ -7 & 5 & -4\end{bmatrix}$ $1+1\frac{1}{2}+\frac{1}{2}$

The system of equations in Matrix form can be written as :

$$A \cdot X = B \text{, where } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$ **1**

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ -35 \\ -35 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -5 \end{bmatrix}$$ **1**

$\therefore$ $x = 1, y = -5, z = -5$ **1**

OR

Using properties of determinants, prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

Ans. $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$

$$= \begin{vmatrix} b^2+c^2 & a^2 & bc \\ c^2+a^2 & b^2 & ca \\ a^2+b^2 & c^2 & ab \end{vmatrix} \qquad (C_1 \to C_1 - 2C_3)$$ **1**

$$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix} \qquad (C_1 \to C_1 + C_2)$$ **1**

$$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ 0 & b^2-a^2 & ca-bc \\ 0 & c^2-a^2 & ab-bc \end{vmatrix} \qquad (R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$$ **2**

$$= (b-a)(c-a)\begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ 0 & b+a & -c \\ 0 & c+a & -b \end{vmatrix}$$ **1**

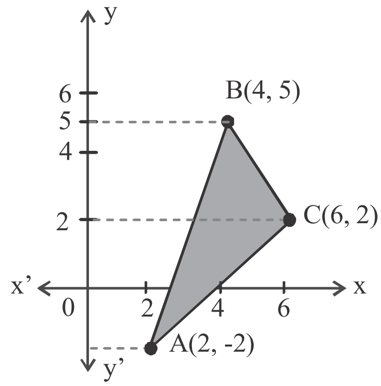
Expand along C_1

$$= \left(a^2+b^2+c^2\right)(b-a)\,(c-a)\left(-b^2-ab+c^2+ac\right)$$

$$= (a-b)\,(b-c)\,(c-a)\,(a+b+c)\left(a^2+b^2+c^2\right)$$ **1**

34. Using integration, find the area of the region bounded by the triangle whose vertices are (2, –2), (4,5) and (6,2).

Ans.



Let A(2, –2) ; B(4, 5) ; C(6, 2)

Equations of the lines

$$\left.\begin{aligned} AB &: x = \frac{2}{7}(y+9) \\ BC &: x = -\frac{2}{3}(y-11) \\ AC &: x = y+4 \end{aligned}\right]$$ **$1\frac{1}{2}$**

Correct graph **1/2**

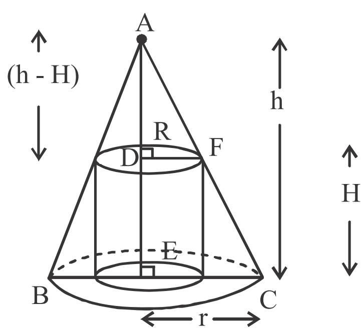
$$ar(\Delta ABC) = \int_{-2}^{2}(y+4)dy + \left(\frac{-2}{3}\right)\int_{2}^{5}(y-11)dy - \int_{-2}^{5}\frac{2}{7}(y+9)\,dy$$ **2**

$$= \frac{1}{2}\left[(y+4)^2\right]_{-2}^{2} - \frac{1}{3}\left[(y-11)^2\right]_{2}^{5} - \frac{1}{7}\left[(y+9)^2\right]_{-2}^{5}$$ **$1\frac{1}{2}$**

$$= 16 + 15 - 21 = 10$$ **1/2**

35. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone.

Ans.



Let H = Height of cylinder

R = Radius of cylinder

Volume of cone $= \frac{\pi}{3}r^2h$ **1/2**

V = Volume of cylinder $= \pi R^2 H$ **1/2**

$$\Delta ADF \sim \Delta AEC \Rightarrow \frac{h-H}{h} = \frac{R}{r} \Rightarrow R = \frac{r}{R}(h-H)$$ **1**

$$\therefore V = \pi \cdot H \cdot \frac{r^2}{h^2}(h-H)^2 = \frac{\pi r^2}{h^2}\left(H^3 - 2hH^2 + Hh^2\right)$$ **1**

$$V'(H) = \frac{\pi r^2}{h^2}\left(3H^2 - 4hH + h^2\right),\ V'(h) = 0 \Rightarrow H = \frac{h}{3}$$ **1+1**

$$V''(H) = \frac{\pi r^2}{h^2}(6H - 4h),\ V''\left(H = \frac{h}{3}\right) = \frac{\pi r^2}{h^2}(-2h) < 0$$ **1/2**

$$\therefore\ V \text{ is max iff } H = \frac{h}{3} \text{ and } R = \frac{2r}{3}$$

$$\frac{\text{Volume of cylinder}}{\text{Volume of cone}} = \frac{3\pi R^2 H}{\pi r^2 h} = 3 \cdot \frac{4r^2}{9} \cdot \frac{h}{3} \cdot \frac{1}{r^2 h} = \frac{4}{9}$$ **1/2**

36. Find the equation of the plane that contains the point A(2,1,–1) and is perpendicular to the line of intersection of the planes $2x + y - z = 3$ and $x + 2y + z = 2$. Also find the angle between the plane thus obtained and the y-axis.

Ans. Let equation of the required plane be:

$a(x - 2) + b(y - 1) + c(z + 1) = 0$ **$1\frac{1}{2}$**

Also : $2a + b - c = 0$

$a + 2b + c = 0$

Solving: $\frac{a}{3} = \frac{b}{-3} = \frac{c}{3} = k \Rightarrow a = 3k, b = -3k, c = 3k$ **$1\frac{1}{2}$**

$\therefore$ Equation of plane is : $3k(x - 2) - 3k(y - 1) + 3k(z + 1) = 0$

$\Rightarrow x - y + z = 0$ **$1\frac{1}{2}$**

Let angle between y-axis and plane = θ

then, $\sin\theta = \left|\frac{0-1+0}{\sqrt{1+1+1}}\right| = \left|\frac{-1}{\sqrt{3}}\right| \Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **$1\frac{1}{2}$**

OR

Find the distance of the point P(–2, –4, 7) from the point of intersection Q of the line $\vec{r} = (3\hat{i} - 2\hat{j} + 6\hat{k}) + \lambda(2\hat{i} - \hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 6$. Also write the vector equation of the line PQ.

Ans. General point on line is: $\vec{r} = (3 + 2\lambda)\hat{i} + (-2 - \lambda)\hat{j} + (6 + 2\lambda)\hat{k}$ **1**

For the point of intersection:

$\left[(3 + 2\lambda)\hat{i} + (-2 - \lambda)\hat{j} + (6 + 2\lambda)\hat{k}\right] \cdot \left(\hat{i} - \hat{j} + \hat{k}\right) = 6$ **1**

$\Rightarrow 3 + 2\lambda + 2 + \lambda + 6 + 2\lambda = 6 \Rightarrow \lambda = -1$ **1**

$\therefore Q(\hat{i} - \hat{j} + 4\hat{k}) = Q(1, -1, 4)$ **1**

$PQ = 3\sqrt{3}$, equation of the line PQ : $\vec{r} = -2\hat{i} - 4\hat{j} + 7\hat{k} + \mu\left(3\hat{i} + 3\hat{j} - 3\hat{k}\right)$ **1+1**

QUESTION PAPER CODE 65/1/2
EXPECTED ANSWER/VALUE POINTS
SECTION – A

Question Numbers 1 to 20 carry 1 mark each.

Question Numbers 1 to 10 are multiple choice type questions. Select the correct option.

Q.No.		Marks
1.	If A is a 3 × 3 matrix and $\|A\| = -2$, then value of $\|A\,(\text{adj } A)\|$ is **(A)** -2 **(B)** 2 **(C)** -8 **(D)** 8	
	Ans: (C) -8	1
2.	The number of arbitrary constants in the particular solution of a differential equation of second order is (are) **(A)** 0 **(B)** 1 **(C)** 2 **(D)** 3	
	Ans: (A) 0	1
3.	The principal value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ **(A)** $\frac{13\pi}{6}$ **(B)** $\frac{\pi}{2}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{6}$	
	Ans: (D) $\frac{\pi}{6}$	1
4.	The corner points of the feasible region determined by the system of linear inequalities are (0, 0), (4, 0), (2, 4) and (0, 5). If the maximum value of $z = ax + by$, where $a, b > 0$ occurs at both (2, 4) and (4, 0), then **(A)** $a = 2b$ **(B)** $2a = b$ **(C)** $a = b$ **(D)** $3a = b$	
	Ans: (A) $a = 2b$	1
5.	If A and B are two independent events with $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, then $P(B' \mid A)$ is equal to **(A)** $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3}{4}$ **(D)** 1	
	Ans: (C) $\frac{3}{4}$	1
6.	If A is a square matrix such that $A^2 = A$, then $(I - A)^3 + A$ is equal to **(A)** I **(B)** 0 **(C)** $I - A$ **(D)** $I + A$	
	Ans: (A) I	1

7. $\int_{-\frac{x}{2}}^{\frac{x}{2}} \frac{1}{x^2} \sin\left(\frac{1}{x}\right) dx$, where $x \neq 0$ is equal to

(A) -2 **(B)** 0 **(C)** 1 **(D)** π

Ans: (B) 0 — **1**

8. The image of the point (2, –1, 5) in the plane $\vec{r} \cdot \hat{i} = 0$ is

(A) (–2, –1, 5) **(B)** (2, 1, –5) **(C)** (–2, 1, –5) **(D)** (2, 0, 0)

Ans: (A) (–2, –1, 5) — **1**

9. If the projection of $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ on $\vec{b} = 2\hat{i} + \lambda\hat{k}$ is zero, then the value of λ is

(A) 0 **(B)** 1 **(C)** $\frac{-2}{3}$ **(D)** $\frac{-3}{2}$

Ans: (C) $\frac{-2}{3}$ — **1**

10. The vector equation of the line passing through the point (–1 , 5, 4) and perpendicular to the plane z = 0 is

(A) $\vec{r} = -\hat{i} + 5\hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j})$ **(B)** $\vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}$

(C) $\vec{r} = \hat{i} - 5\hat{j} - 4\hat{k} + \lambda\hat{k}$ **(D)** $\vec{r} = \lambda\hat{k}$

Ans: (B) $\vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}$ — **1**

Fill in the blanks in questions numbers 11 to 15

11. The position vectors of two points A and B are $\overrightarrow{OA} = 2\hat{i} - \hat{j} - \hat{k}$ and $\overrightarrow{OB} = 2\hat{i} - \hat{j} + 2\hat{k}$, respectivley. The position vector of a point P which divides the line segment joining A and B in the ratio 2 : 1 is __________.

Ans: $2\hat{i} - \hat{j} + \hat{k}$ — **1**

12. The equation of the normal to the curve $y^2 = 8x$ at the origin is __________.

Ans: $y = 0$ — **1**

OR

The radius of a circle is increasing at the uniform rate of 3 cm/sec. At the instant when the radius of the circle is 2 cm, its area increases at the rate of __________ cm^2/s.

Ans: 12π — **1**

13. On applying elementary column operation $C_2 \to C_2 - 3C_1$ in the matriax equation $\begin{bmatrix} 4 & -2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, the RHS (Right Hand Side) of the equation becomes ___________.

Ans: $\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 2 & -6 \end{bmatrix}$ **1**

OR

A square matrix A is said to be symmetric if ________

Ans: $A = A'$ **1**

14. A relation R in a set A is called _________, if $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

Ans: Symmetric **1**

15. The greatest integer fucntion defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at $x =$ ______.

Ans: 1 **1**

Question numbers 16 to 20 are very short answer type questions

16. If A is non-singular square matrix of order 3 and $A^2 = 2A$, then find the value of $|A|$.

Ans: $|A|^2 = 8|A|$ **1/2**

$\Rightarrow |A| = 8$ **1/2**

17. Two cards are drawn at random and one-by-one without replacement from a wll-shuffled pack of 52 playing cards. Find the probability that one card is red and the other is black.

Ans: $\dfrac{{}^{26}C_1 \times {}^{26}C_1}{{}^{52}C_2} = \dfrac{26}{51}$ **1/2+1/2**

18. Evaluate $\int_1^3 |2x - 1|\,dx$.

Ans: $\int_1^3 |2x-1|\,dx = \int_1^3 (2x-1)\,dx = \left[\frac{1}{4}(2x-1)^2\right]_1^3$ **1/2**

$= 6$ **1/2**

19. Find : $\int \frac{dx}{\sqrt{9-4x^2}}$

Ans: $\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{dx}{\sqrt{3^2-(2x)^2}}$ **1/2**

$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$ **1/2**

20. Find: $\int x^4 \log x \, dx$.

Ans: $\int x^4 \cdot \log x dx = \log x \cdot \frac{x^5}{5} - \int \frac{1}{x} \cdot \frac{x^5}{5} dx$ **1/2**

$= \frac{x^5 \cdot \log x}{5} - \frac{x^5}{25} + c$ **1/2**

OR

Find: $\int \frac{2x}{\sqrt[3]{x^2+1}} dx$.

Ans: Let, $x^2 + 1 = t$

$\therefore \quad 2xdx = dt$ **1/2**

$$\int \frac{2x}{\sqrt[3]{x^2+1}} dx = \int \frac{1}{\sqrt[3]{t}} dt = \int t^{-1/3} dt = \frac{3}{2} t^{2/3} + c$$

$= \frac{3}{2}\left(x^2+1\right)^{2/3} + c$ **1/2**

SECTION-B

Question numbers 21 to 26 carry 2 marks each.

21. Find a unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$ where $\vec{a} = 5\hat{i} + 6\hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 6\hat{j} + 2\hat{k}$.

Ans: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 6 & -2 \\ 7 & 6 & 2 \end{vmatrix} = 24\hat{i} - 24\hat{j} - 12\hat{k}$ **1**

Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$ **1**

OR

Find the volume of the parallelopiped whose adjacent edges are represented by $2\vec{a}, -\vec{b}$ and $3\vec{c}$, where $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 3\hat{k}$

Ans: Volume of the parallelopiped = $\begin{vmatrix} 2 & -2 & 4 \\ -3 & -4 & 5 \\ 6 & -3 & 9 \end{vmatrix}$ **1**

$= |-24| = 24$ **1**

22. Examine the applicability of Rolle's theorem for the function f(x) = sin 2x in [0, π]. Hence find the points where the tangent is parallel to x-axis.

Ans: As, sine function and polynomial function are everywhere continuous and differentiable.

∴ (i) f(x) = sin 2x is continuous on [0, π]
(ii) f(x) = sin 2x is differentiable on (0, π)
(iii) f(0) = 0 = f(π) **1**

∴ Rolle's Theorem is applicable for f(x) = sin2x

Solving, $f'(x) = 0$ or $2\cos 2x = 0 \Rightarrow \cos 2x = 0$

∴ $2x = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$ **1/2**

The points where the tangent is parallel to x-axis are: $\left(\frac{\pi}{4}, 1\right); \left(\frac{3\pi}{4}, -1\right)$ **1/2**

23. Find the values of x for which the fucntion f(x) = 2 + 3x – x^3 is decreasing.

Ans: f(x) is decreasing iff $f'(x) \le 0$ **1**

$\Leftrightarrow 3 - 3x^2 \le 0$ or $x^2 \ge 1$

$\Leftrightarrow x \le -1$ or $x \ge 1$ **1**

24. The probability of finding a green signal on a busy crossing X is 30%. What is the probability of finding a green signal on X on two consecutive days out of three?

Ans: Probability of green signal on crossing $X = \frac{30}{100} = \frac{3}{10}$ **1**

Probability of not a green signal on crossing $X = 1 - \frac{3}{10} = \frac{7}{10}$

Probability of a green signal on X on two concecutative days out of three

$= \frac{3}{10} \times \frac{3}{10} \times \frac{7}{10} + \frac{7}{10} \times \frac{3}{10} \times \frac{3}{10} = \frac{63}{500}$ **1**

25. Prove that $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x, \frac{1}{\sqrt{2}} \le x \le 1$

Ans: Put $x = \cos\theta \Leftrightarrow \theta = \cos^{-1}x$ **1/2**

L.H.S. $= \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$= \sin^{-1}(2\cos\theta\,\sin\theta) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\cos^{-1}x = \text{R.H.S.}$ **$1\frac{1}{2}$**

OR

Consider a bijective function $f: R_+ \to (7,\infty)$ given by $f(x) = 16x^2 + 24x + 7$, where R_+ is the set of all positive real numbers. Find the inverse function of f.

Ans: Let $y = f(x) = 16x^2 + 24x + 7 = (4x+3)^2 - 2$ **1**

$\Rightarrow f^{-1}(y) = x = \dfrac{\sqrt{y+2}-3}{4}$ **1**

26. Find the value of k so that the lines $x = -y = kz$ and $x - 2 = 2y + 1 = -z + 1$ are perpendicular to each other.

Ans: The lines, $\dfrac{x}{1} = \dfrac{y}{-1} = \dfrac{z}{\frac{1}{k}}$ and $\dfrac{x-2}{1} = \dfrac{y+\frac{1}{2}}{\frac{1}{2}} = \dfrac{z-1}{-1}$ **1**

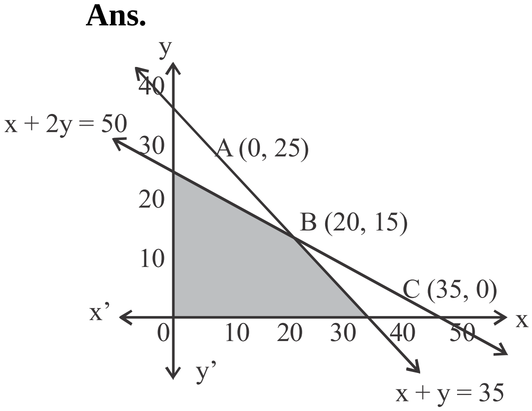
are perpendicular $\therefore\ 1 - \dfrac{1}{2} - \dfrac{1}{k} = 0 \Rightarrow k = 2$ **1**

SECTION-C

Question numbers 27 to 32 carry 4 marks each.

27. A furniture trader deals in only two items – chairs and tables. He has ₹ 50,000 to invest and a space to store at most 35 items. A chair costs him ₹ 1000 and a table costs him ₹ 2000. The trader earns a profit of ₹ 150 and ₹ 250 on a chair and table, respectively. Formulate the above problem as an LPP to maximise the profit and solve it graphically.

Ans.



Let No. of chairs = x, No. of tables = y

Then L.P. P. is:

Maximize (Profit) : $Z = 150x + 250y$ **1**

$$\left.\begin{array}{l} \text{Subject to: } x + y \le 35 \\ 1000x + 2000y \le 50000 \Rightarrow x + 2y \le 50 \\ x, y \ge 0 \end{array}\right\}$$ **1**

Correct graph **$1\frac{1}{2}$**

Corner:	Value of Z	
A(0, 25)	₹ 6250	
B(20, 15)	₹ 6750 (Max)	
C(35, 0)	₹ 5250	1/2

$\therefore$ Max (Z) = ₹ 6750

Number of chairs = 20, Tables = 15

28. If $x = a\sec^3\theta$, $y = a\tan^3\theta$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$.

Ans. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \sin\theta$ **2**

$\frac{d^2y}{dx^2} = \cos\theta\cdot\frac{d\theta}{dx} = \frac{1}{3a\sec^4\theta\cdot\tan\theta}$ **$1\frac{1}{2}$**

$\left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{4}} = \frac{1}{12a}$ **1/2**

29. Find: $\int \frac{2x+1}{\sqrt{3+2x-x^2}}dx$.

Ans. $\int \frac{2x+1}{\sqrt{3+2x-x^2}}dx = -\int \frac{2-2x}{\sqrt{3+2x-x^2}}dx + 3\int \frac{1}{\sqrt{2^2-(x-1)^2}}dx$ **2**

$= -2\sqrt{3+2x-x^2} + 3\sin^{-1}\left(\frac{x-1}{2}\right) + c$ **2**

30. There are two bags, I and II. Bag I contains 3 red and 5 black balls and Bag II contains 4 red and 3 black balls. One ball is transferred randomly from Bag I to Bag II and then a ball is drawn randomly from Bag II. If the ball so drawn is found to be black in colour, then find the probability that the transferred ball is also black.

Ans. E_1 = Event that the ball transfered from Bag I is Black

E_2 = Event that the ball transfered from Bag I is Red **1/2**

A = Event that the ball drawn from Bag II is Black

$P(E_1) = \frac{5}{8}; P(E_2) = \frac{3}{8}; P\left(\frac{A}{E_1}\right) = \frac{4}{8} = \frac{1}{2}; P\left(\frac{A}{E_2}\right) = \frac{3}{8}$ **2**

Required Probability:

$$P\left(\frac{E_1}{A}\right)=\frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right)+P(E_2)\cdot P\left(\frac{A}{E_2}\right)}=\frac{\frac{5}{8}\cdot\frac{1}{2}}{\frac{5}{8}\cdot\frac{1}{2}+\frac{3}{8}\cdot\frac{3}{8}}=\frac{20}{29}$$ $1\frac{1}{2}$

OR

An urn contains 5 red, 2 white and 3 black balls. Three balls are drawn, one-by-one, at random without replacement. Find the probability distribution of the number of white balls. Also, find the mean and the variance of the number of white balls drawn.

Ans. Let X = No. of white balls = 0, 1, 2

X :	0	1	2	1/2
P(X) :	$\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}=\frac{7}{15}$	$3\times\frac{8}{10}\times\frac{7}{9}\times\frac{2}{8}=\frac{7}{15}$	$3\times\frac{2}{10}\times\frac{1}{9}\times\frac{8}{8}=\frac{1}{15}$	$1\frac{1}{2}$
$X\cdot P(X)$:	0	$\frac{7}{15}$	$\frac{2}{15}$	1/2
$X^2P(X)$:	0	$\frac{7}{15}$	$\frac{4}{15}$	

Mean $=\sum XP(X)=\frac{9}{15}=\frac{3}{5}$ 1/2

Variance $=\sum X^2P(x)-\left[\sum X\,P(X)^2\right]=\frac{11}{15}-\left[\frac{3}{5}\right]^2=\frac{28}{75}$ 1

31. Find the general solution of the differential equation $ye^y dx=(y^3+2xe^y)dy$.

Ans. $y\cdot e^y dx=\left(y^3+2xe^y\right)dy\Rightarrow y\cdot e^y\frac{dy}{dx}=y^3+2xe^y$

$\therefore\ \frac{dx}{dy}-\frac{2}{y}x=y^2\cdot e^{-y}$ 1

I.F. (Integrating factor) $=e^{-2\int\frac{1}{y}dy}=e^{-2\log y}=e^{\log\frac{1}{y^2}}=\frac{1}{y^2}$ 1

$\therefore$ Solution is

$$x\cdot\frac{1}{y^2}=\int y^2\cdot e^{-y}\cdot\frac{1}{y^2}dy+c=\int e^{-y}dy+c$$ 1

$\Rightarrow\ \frac{x}{y^2}=-e^{-y}+c$ or $x=-y^2e^{-y}+cy^2$ 1

OR

Find the particular solution of the differential equation

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right), \text{ given that } y = \frac{\pi}{4} \text{ at } x = 1.$$

Ans. The differential equation can be written as:

$$\frac{dy}{dx} = \frac{y}{x} - \tan\frac{y}{x}, \text{ let } y = vx \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$ **1**

$$\Rightarrow v + x\frac{dv}{dx} = v - \tan v \Rightarrow \cot v\, dv = -\frac{1}{x}dx$$

Integrate both sides

$$\log\sin v = -\log|x| + \log c \Rightarrow \log\sin\frac{y}{x} = \log\frac{c}{x}$$ **2**

$$\Rightarrow x\cdot\sin\frac{y}{x} = c, \text{ Put } y = \frac{\pi}{4} \text{ and } x = 1$$

$$\Rightarrow \sin\frac{\pi}{4} = c \text{ or } c = \frac{1}{\sqrt{2}}$$ **1/2**

$\therefore$ Particular solution is $x\cdot\sin\left(\frac{y}{x}\right) = \frac{1}{\sqrt{2}}$ **1/2**

32. Let N be the set of natural numbers and R be the relation on N × N defined by (a, b) R (c, d) iff ad = bc for all a, b, c, d ∈ N. Show that R is an equivalence relation.

Ans: Reflexive: For any (a, b) ∈ N × N

$a\cdot b = b\cdot a$

$\therefore$ (a, b) R (a, b) thus R is reflexive **1**

Symmetric: For (a, b), (c, d) ∈ N × N

(a, b) R (c, d) $\Rightarrow a\cdot d = b\cdot c$

$\Rightarrow c\cdot b = d\cdot a$

$\Rightarrow$ (c, d) R (a, b) $\therefore$ R is symmetric **$1\frac{1}{2}$**

Transitive : For any (a, b), (c, d), (e, f), ∈ N × N

(a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow a\cdot d = b\cdot c$ and $c\cdot f = d\cdot e$

$\Rightarrow a\cdot d\cdot c\cdot f = b\cdot c\cdot d\cdot e \Rightarrow a\cdot f = b\cdot e$

$\therefore$ (a, b) R (e, f), $\therefore$ R is transitive **$1\frac{1}{2}$**

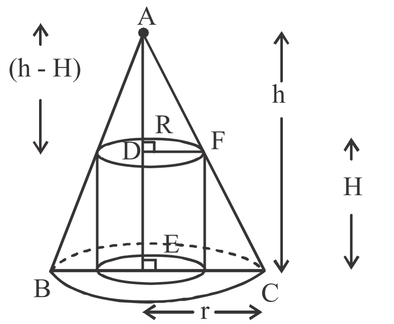
$\therefore$ R is an equivalance Relation

SECTION-D

Question numbers 33 to 36 carry 6 marks.

33. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone.

Ans.



Let H = Height of cylinder

R = Radius of cylinder

Volume of cone $= \frac{\pi}{3} r^2 h$ **1/2**

V = Volume of cylinder $= \pi R^2 H$ **1/2**

$\Delta ADF \sim \Delta AEC \Rightarrow \frac{h-H}{h} = \frac{R}{r} \Rightarrow R = \frac{r}{R}(h-H)$ **1**

$\therefore V = \pi \cdot H \cdot \frac{r^2}{h^2}(h-H)^2 = \frac{\pi r^2}{h^2}\left(H^3 - 2hH^2 + Hh^2\right)$ **1**

$V'(H) = \frac{\pi r^2}{h^2}\left(3H^2 - 4hH + h^2\right),\ V'(h) = 0 \Rightarrow H = \frac{h}{3}$ **1+1**

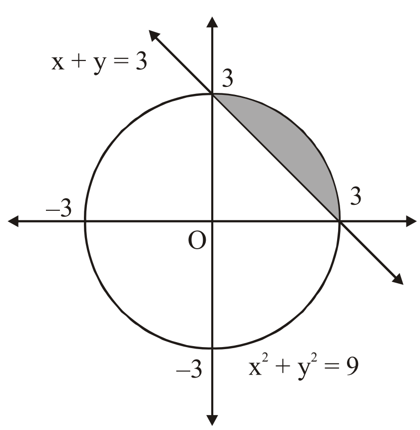
$V''(H) = \frac{\pi r^2}{h^2}(6H - 4h),\ V''\left(H = \frac{h}{3}\right) = \frac{\pi r^2}{h^2}(-2h) < 0$ **1/2**

$\therefore$ V is max iff $H = \frac{h}{3}$ and $R = \frac{2r}{3}$

$$\frac{\text{Volume of cylinder}}{\text{Volume of cone}} = \frac{3\pi R^2 H}{\pi r^2 h} = 3 \cdot \frac{4r^2}{9} \cdot \frac{h}{3} \cdot \frac{1}{r^2 h} = \frac{4}{9}$$ **1/2**

34. Using integration, find the area of the region $\{(x, y) : x^2 + y^2 \le 9,\ x + y \ge 3\}$

Ans. Correct graph. **2**



Required area

$= \int_0^3 \sqrt{9-x^2}\,dx - \int_0^3 (3-x)\,dx$ **2**

$= \left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 + \left[\frac{1}{2}(3-x)^2\right]_0^3$ **$1\frac{1}{2}$**

$= \frac{9\pi}{4} - \frac{9}{2}$ or $\frac{9}{4}(\pi - 2)$ **1/2**

35. Find the equation of the plane that contains the point A(2,1,–1) and is perpendicular to the line of intersection of the planes $2x+y-z=3$ and $x+2y+z=2$. Also find the angle between the plane thus obtained and the y-axis.

Ans. Let equation of the required plane be:

$a(x-2)+b(y-1)+c(z+1)=0$ $1\frac{1}{2}$

Also : $2a+b-c=0$

$a+2b+c=0$

Solving: $\frac{a}{3}=\frac{b}{-3}=\frac{c}{3}=k \Rightarrow a=3k,\ b=-3k,\ c=3k$ $1\frac{1}{2}$

$\therefore$ Equation of plane is : $3k(x-2)-3k(y-1)+3k(z+1)=0$

$\Rightarrow x-y+z=0$ $1\frac{1}{2}$

Let angle between y-axis and plane = θ

then, $\sin\theta=\left|\frac{0-1+0}{\sqrt{1+1+1}}\right|=\left|\frac{-1}{\sqrt{3}}\right|\Rightarrow\theta=\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $1\frac{1}{2}$

OR

Find the distance of the point P(–2, –4, 7) from the point of intersection Q of the line $\vec{r}=(3\hat{i}-2\hat{j}+6\hat{k})+\lambda(2\hat{i}-\hat{j}+2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=6$. Also write the vector equation of the line PQ.

Ans. General point on line is: $\vec{r}=(3+2\lambda)\hat{i}+(-2-\lambda)\hat{j}+(6+2\lambda)\hat{k}$ 1

For the point of intersection:

$\left[(3+2\lambda)\hat{i}+(-2-\lambda)\hat{j}+(6+2\lambda)\hat{k}\right]\cdot\left(\hat{i}-\hat{j}+\hat{k}\right)=6$ 1

$\Rightarrow 3+2\lambda+2+\lambda+6+2\lambda=6\Rightarrow\lambda=-1$ 1

$\therefore Q(\hat{i}-\hat{j}+4\hat{k})=Q\,(1,-1,4)$ 1

$PQ=3\sqrt{3}$, equation of the line PQ : $\vec{r}=-2\hat{i}-4\hat{j}+7\hat{k}+\mu\left(3\hat{i}+3\hat{j}-3\hat{k}\right)$ 1+1

36. If $A=\begin{bmatrix}1 & 2 & -3\\ 3 & 2 & -2\\ 2 & -1 & 1\end{bmatrix}$, then find A^{-1} and use it to solve the following system of the equations:

$x+2y-3z=6$

$3x+2y-2z=3$

$2x-y+z=2$

Ans. $|A| = 7$; $\text{adj}(A) = \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}$; $A^{-1} = \frac{1}{|A|}\text{adj}\, A = \frac{1}{7}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}$ **$1+1\frac{1}{2}+\frac{1}{2}$**

The system of equations in Matrix form can be written as :

$A \cdot X = B$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$ **1**

$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ -35 \\ -35 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -5 \end{bmatrix}$ **1**

$\therefore\ x = 1, y = -5, z = -5$ **1**

OR

Using properties of determinants, prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

Ans. $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$

$= \begin{vmatrix} b^2+c^2 & a^2 & bc \\ c^2+a^2 & b^2 & ca \\ a^2+b^2 & c^2 & ab \end{vmatrix}$ $(C_1 \to C_1 - 2C_3)$ **1**

$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix}$ $(C_1 \to C_1 + C_2)$ **1**

$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ 0 & b^2-a^2 & ca-bc \\ 0 & c^2-a^2 & ab-bc \end{vmatrix}$ $(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$ **2**

$$= (b-a)(c-a)\begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ 0 & b+a & -c \\ 0 & c+a & -b \end{vmatrix}$$ **1**

Expand along C_1

$$= \left(a^2+b^2+c^2\right)(b-a)\,(c-a)\left(-b^2-ab+c^2+ac\right)$$

$$= (a-b)\,(b-c)\,(c-a)\,(a+b+c)\left(a^2+b^2+c^2\right)$$ **1**

QUESTION PAPER CODE 65/1/3

EXPECTED ANSWER/VALUE POINTS

SECTION – A

Question Numbers 1 to 20 carry 1 mark each.

Question Numbers 1 to 10 are multiple choice type questions. Select the correct option.

Q.No. **Marks**

1. The matrix $\begin{bmatrix} 2 & -1 & 3 \\ \lambda & 0 & 7 \\ -1 & 1 & 4 \end{bmatrix}$ is not invertible for

(A) $\lambda = -1$ **(B)** $\lambda = 0$ **(C)** $\lambda = 1$ **(D)** $\lambda \in R - \{1\}$

Ans: (C) $\lambda = 1$ **1**

2. The number of arbitrary constants in the particular solution of a differential equation of second order is (are)

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3

Ans: (A) 0 **1**

3. The value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ is

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{2}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{7\pi}{6}$

Ans: (A) $\frac{\pi}{6}$ **1**

4. The corner points of the feasible retgion determined by the system of linear inequalities are (0, 0), (4, 0), (2, 4) and (0, 5). If the maximum value of $z = ax + by$, where $a, b > 0$ occurs at both (2, 4) and (4, 0), then

(A) $a = 2b$ **(B)** $2a = b$ **(C)** $a = b$ **(D)** $3a = b$

Ans: (A) $a = 2b$ **1**

5. If A and B are two independent events with $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, then $P(B' \mid A)$ is equal to

(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3}{4}$ **(D)** 1

Ans: (C) $\frac{3}{4}$ **1**

6. If A is a square matrix such that $A^2 = A$, then $(I - A)^3 + A$ is equal to

(A) I **(B)** 0 **(C)** $I - A$ **(D)** $I + A$

Ans: (A) I **1**

7. $\int_1^e \frac{\log x}{x} dx$, is equal to

(A) $\frac{e^2}{2}$ **(B)** 1 **(C)** $\frac{1}{2}$ **(D)** $-\infty$

Ans: (C) $\frac{1}{2}$ **1**

8. A point P lies on the line segment joining the points (–1, 3, 2) and (5, 0, 6). If x-coordinate of P is 2, then its z-coordinate is

(A) –1 **(B)** 4 **(C)** $\frac{3}{2}$ **(D)** 8

Ans: (B) 4 **1**

9. If the projection of $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ on $\vec{b} = 2\hat{i} + \lambda\hat{k}$ is zero, then the value of λ is

(A) 0 **(B)** 1 **(C)** $\frac{-2}{3}$ **(D)** $\frac{-3}{2}$

Ans: (C) $\frac{-2}{3}$ **1**

10. The vector equation of the line passing through the point (–1, 5, 4) and perpendicular to the plane z = 0 is

(A) $\vec{r} = -\hat{i} + 5\hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j})$ **(B)** $\vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}$

(C) $\vec{r} = \hat{i} - 5\hat{j} - 4\hat{k} + \lambda\hat{k}$ **(D)** $\vec{r} = \lambda\hat{k}$

Ans: (B) $\vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}$ **1**

Fill in the blanks in question numbers 11 to 15

11. The position vectors of two points A and B are $\overrightarrow{OA} = 2\hat{i} - \hat{j} - \hat{k}$ and $\overrightarrow{OB} = 2\hat{i} - \hat{j} + 2\hat{k}$, respectivley. The position vector of a point P which divides the line segment joining A and B in the ratio 2 : 1 is __________.

Ans: $2\hat{i} - \hat{j} + \hat{k}$ **1**

12. The equation of the normal to the curve $y^2 = 8x$ at the origin is __________.

Ans: y = 0 **1**

OR

The radius of a circle is increasing at the uniform rate of 3 cm/sec. At the instant when the radius of the circle is 2 cm, its area increases at the rate of __________ cm^2/s.

Ans: 12π **1**

13. If A is a square matrix of order 3 and A_{ij} is the cofactor of the element a_{ij}, then value of $a_{21} A_{11} + a_{22} A_{12} + a_{23} A_{13}$ is __________.

Ans: 0 **1**

OR

If the matrix A is both symmetric and skew symmetric, then A is a __________.

Ans: Zero matrix **1**

14. A relation R in a set A is called ________, if $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

Ans: Symmetric **1**

15. The greatest integer fucntion defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at x = ______.

Ans: 1 **1**

Question numbers 16 to 20 are very short answer type questions

16. If A is a square matrix of order 3 and |A| = 2, then find the value of | –AA' |.

Ans: $|-AA'| = -|A|^2$ **1/2**

$= -4$ **1/2**

17. Two cards are drawn at random and one-by-one without replacement from a well-shuffled pack of 52 playing cards. Find the probability that one card is red and the other is black.

Ans: $\dfrac{{}^{26}C_1 \times {}^{26}C_1}{{}^{52}C_2} = \dfrac{26}{51}$ **1/2+1/2**

18. Evaluate: $\int_1^3 |2x - 1|\,dx$.

Ans: $\int_1^3 |2x-1|\,dx = \int_1^3 (2x-1)\,dx = \left[\dfrac{1}{4}(2x-1)^2\right]_1^3$ **1/2**

$= 6$ **1/2**

19. Find : $\int \dfrac{dx}{\sqrt{9 - 4x^2}}$

Ans: $\int \dfrac{dx}{\sqrt{9-4x^2}} = \int \dfrac{dx}{\sqrt{3^2 - (2x)^2}}$ **1/2**

$= \dfrac{1}{2}\sin^{-1}\left(\dfrac{2x}{3}\right) + C$ **1/2**

20. Find: $\int x^4 \log x \, dx$.

Ans: $\int x^4 \cdot \log x dx = \log x \cdot \frac{x^5}{5} - \int \frac{1}{x} \cdot \frac{x^5}{5} dx$ 1/2

$= \frac{x^5 \cdot \log x}{5} - \frac{x^5}{25} + c$ 1/2

OR

Find: $\int \frac{2x}{\sqrt[3]{x^2+1}} dx$.

Ans: Let, $x^2 + 1 = t$

$\therefore \quad 2x dx = dt$ 1/2

$\int \frac{2x}{\sqrt[3]{x^2+1}} dx = \int \frac{1}{\sqrt[3]{t}} dt = \int t^{-1/3} dt = \frac{3}{2} t^{2/3} + c$

$= \frac{3}{2}\left(x^2+1\right)^{2/3} + c$ 1/2

SECTION-B

Question numbers 21 to 26 carry 2 marks each.

21. Find a unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$ where $\vec{a} = 5\hat{i} + 6\hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 6\hat{j} + 2\hat{k}$.

Ans: $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 6 & -2 \\ 7 & 6 & 2 \end{vmatrix} = 24\hat{i} - 24\hat{j} - 12\hat{k}$ 1

Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$ 1

OR

Find the volume of the parallelopiped whose adjacent edges are represented by $2\vec{a}, -\vec{b}$ and $3\vec{c}$, where $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 3\hat{k}$

Ans: Volume of the parallelopiped $= \begin{vmatrix} 2 & -2 & 4 \\ -3 & -4 & 5 \\ 6 & -3 & 9 \end{vmatrix}$ 1

$= |-24| = 24$ 1

22. If $f(x) = \sqrt{\tan\sqrt{x}}$, then find $f'\left(\frac{\pi^2}{16}\right)$.

Ans: $f'(x) = \dfrac{\sec^2\sqrt{x}}{4\sqrt{x}\sqrt{\tan(\sqrt{x})}}$ **1**

$f'\left(\frac{\pi}{16}\right) = \dfrac{2}{4\cdot\frac{\pi}{4}} = \dfrac{2}{\pi}$ **1**

23. Using differentials, find the approximate value of $\sqrt{25\cdot 3}$ up to two places of decimals.

Ans: Let $y = f(x) = \sqrt{x}$, Let $x = 25$, $x + \Delta x = 25.3$, $\Delta x = 0.3$ **1**

$\Delta y \simeq \left.\dfrac{dy}{dx}\right|_{x=25} \cdot \Delta x = \dfrac{1}{2\sqrt{25}}(0.3) = 0.03$ **1/2**

$\sqrt{25.3} = f(25) + \Delta y = 5 + 0.03 = 5.03$ (approx.) **1/2**

24. The probability of finding a green signal on a busy crossing X is 30%. What is the probability of finding a green signal on X on two consecutive days out of three?

Ans: Probability of green signal on crossing $X = \frac{30}{100} = \frac{3}{10}$

Probability of not a green signal on crossing $X = 1 - \frac{3}{10} = \frac{7}{10}$ **1**

Probability of a green signal on X on two concecutative days out of three

$= \frac{3}{10} \times \frac{3}{10} \times \frac{7}{10} + \frac{7}{10} \times \frac{3}{10} \times \frac{3}{10} = \frac{63}{500}$ **1**

25. Prove that $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x, \frac{1}{\sqrt{2}} \le x \le 1$

Ans: Put $x = \cos\theta \Leftrightarrow \theta = \cos^{-1}x$ **1/2**

L.H.S. $= \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$= \sin^{-1}(2\cos\theta\,\sin\theta) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\cos^{-1}x =$ R.H.S. **$1\frac{1}{2}$**

OR

Consider a bijective function $f: R_+ \rightarrow (7, \infty)$ given by $f(x) = 16x^2 + 24x + 7$, where R_+ is the set of all positive real numbers. Find the inverse function of f.

Ans: Let $y = f(x) = 16x^2 + 24x + 7 = (4x + 3)^2 - 2$ **1**

$$\Rightarrow f^{-1}(y) = x = \frac{\sqrt{y+2} - 3}{4}$$ **1**

26. Find the value of k so that the lines $x = -y = kz$ and $x - 2 = 2y + 1 = -z + 1$ are perpendicular to each other.

Ans: The lines, $\frac{x}{1} = \frac{y}{-1} = \frac{z}{\frac{1}{k}}$ and $\frac{x-2}{1} = \frac{y+\frac{1}{2}}{\frac{1}{2}} = \frac{z-1}{-1}$ **1**

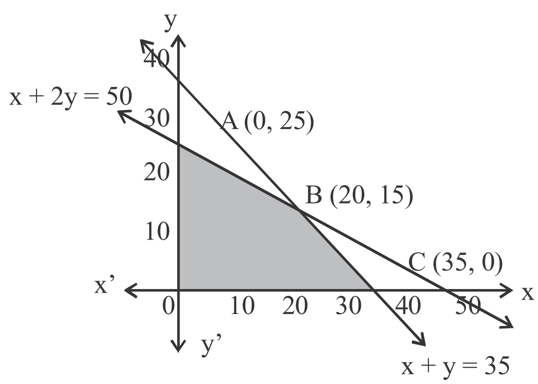
are perpendicular $\therefore\ 1 - \frac{1}{2} - \frac{1}{k} = 0 \Rightarrow k = 2$ **1**

SECTION-C

Question numbers 27 to 32 carry 4 marks each.

27. A furniture trader deals in only two items – chairs and tables. He has ₹ 50,000 to invest and a space to store at most 35 items. A chair costs him ₹ 1000 and a table costs him ₹ 2000. The trader earns a profit of ₹ 150 and ₹ 250 on a chair and table, respectively. Formulate the above problem as an LPP to maximise the profit and solve it graphically.

Ans.



Let No. of chairs = x, No. of tables = y

Then L.P. P. is:

Maximize (Profit) : $Z = 150x + 250y$ **1**

Subject to : $x + y \leq 35$

$1000x + 2000y \leq 50000 \Rightarrow x + 2y \leq 50$ **1**

$x, y \geq 0$

Correct graph **$1\frac{1}{2}$**

Corner:	Value of Z
A(0, 25)	₹ 6250
B(20, 15)	₹ 6750 (Max)
C(35, 0)	₹ 5250

1/2

$\therefore$ Max (Z) = ₹ 6750

Number of chairs = 20, Tables = 15

28. If $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $a > 0$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{3}$.

Ans. $\frac{dy}{d\theta} = a\sin\theta$, $\frac{dx}{d\theta} = a(1 - \cos\theta)$ **1/2+1/2**

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$ **1**

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\cdot\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{d\theta}{dx} = -\frac{\operatorname{cosec}^2\frac{\theta}{2}}{2a(1-\cos\theta)}$$ **1**

$$\left.\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{3}} = -\frac{1}{2}\times\frac{4}{a\left(1-\frac{1}{2}\right)} = -\frac{4}{a}$$ **1**

29. Evaluate $\int_2^3 e^x\,dx$ as limit of the sums.

Ans. Let $f(x) = e^x, a = 1, b = 3, nh = 2$, ... **1**

$$f(1) + f(1+h) + f(1+2h) + ... + f(1+(n-1)h)$$

$$= e + e^{1+h} + e^{1+2h} + ... + e^{1+(n-1)h} = \frac{e(e^{nh}-1)}{e^h - 1}$$ **2**

$$\int_1^3 e^x dx = \lim_{h\to 0} h\cdot\frac{e(e^{nh}-1)}{e^h-1} = e(e^2-1) \text{ or } e^3 - e$$ **1**

30. There are two bags, I and II. Bag I contains 3 red and 5 black balls and Bag II contains 4 red and 3 black balls. One ball is transferred randomly from Bag I to Bag II and then a ball is drawn randomly from Bag II. If the ball so drawn is found to be black in colour, then find the probability that the transferred ball is also black.

Ans. E_1 = Event that the ball transfered from Bag I is Black
E_2 = Event that the ball transfered from Bag I is Red
A = Event that the ball drawn from Bag II is Black **1/2**

$$P(E_1) = \frac{5}{8};\ P(E_2) = \frac{3}{8};\ P\left(\frac{A}{E_1}\right) = \frac{4}{8} = \frac{1}{2};\ P\left(\frac{A}{E_2}\right) = \frac{3}{8}$$ **2**

Required Probability:

$$P\left(\frac{E_1}{A}\right)=\frac{P(E_1)\cdot P\left(\frac{A}{E_1}\right)}{P(E_1)\cdot P\left(\frac{A}{E_1}\right)+P(E_2)\cdot P\left(\frac{A}{E_2}\right)}=\frac{\frac{5}{8}\cdot\frac{1}{2}}{\frac{5}{8}\cdot\frac{1}{2}+\frac{3}{8}\cdot\frac{3}{8}}=\frac{20}{29}$$ $1\frac{1}{2}$

OR

An urn contains 5 red, 2 white and 3 black balls. Three balls are drawn, one-by-one, at random without replacement. Find the probability distribution of the number of white balls. Also, find the mean and the variance of the number of white balls drawn.

Ans. Let X = No. of white balls = 0, 1, 2

X :	0	1	2	1/2
P(X) :	$\frac{8}{10}\times\frac{7}{9}\times\frac{6}{8}=\frac{7}{15}$	$3\times\frac{8}{10}\times\frac{7}{9}\times\frac{2}{8}=\frac{7}{15}$	$3\times\frac{2}{10}\times\frac{1}{9}\times\frac{8}{8}=\frac{1}{15}$	$1\frac{1}{2}$
X · P(X) :	0	$\frac{7}{15}$	$\frac{2}{15}$	1/2
$X^2P(X)$:	0	$\frac{7}{15}$	$\frac{4}{15}$	

Mean = $\sum XP(X)=\frac{9}{15}=\frac{3}{5}$ 1/2

Variance = $\sum X^2P(x)-\left[\sum X\,P(X)^2\right]=\frac{11}{15}-\left[\frac{3}{5}\right]^2=\frac{28}{75}$ 1

31. Find the general solution of the differential equation $ye^y dx=(y^3+2xe^y)dy$.

Ans. $y\cdot e^y dx=\left(y^3+2xe^y\right)dy \Rightarrow y\cdot e^y\frac{dy}{dx}=y^3+2xe^y$

$\therefore\ \frac{dx}{dy}-\frac{2}{y}x=y^2\cdot e^{-y}$ 1

I.F. (Integrating factor) = $e^{-2\int\frac{1}{y}dy}=e^{-2\log y}=e^{\log\frac{1}{y^2}}=\frac{1}{y^2}$ 1

$\therefore$ Solution is

$x\cdot\frac{1}{y^2}=\int y^2\cdot e^{-y}\cdot\frac{1}{y^2}dy+c=\int e^{-y}dy+c$ 1

$\Rightarrow\frac{x}{y^2}=-e^{-y}+c$ or $x=-y^2e^{-y}+cy^2$ 1

OR

Find the particular solution of the differential equation

$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$, given that $y = \frac{\pi}{4}$ at $x = 1$.

Ans. The differential equation can be written as:

$\frac{dy}{dx} = \frac{y}{x} - \tan\frac{y}{x}$, let $y = vx$ $\therefore$ $\frac{dy}{dx} = v + x\frac{dv}{dx}$ **1**

$\Rightarrow$ $v + x\frac{dv}{dx} = v - \tan v$ $\Rightarrow$ $\cot v \, dv = -\frac{1}{x}dx$

Integrate both sides

$\log \sin v = -\log|x| + \log c \Rightarrow \log \sin\frac{y}{x} = \log\frac{c}{x}$ **2**

$\Rightarrow$ $x \cdot \sin\frac{y}{x} = c$, Put $y = \frac{\pi}{4}$ and $x = 1$

$\Rightarrow$ $\sin\frac{\pi}{4} = c$ or $c = \frac{1}{\sqrt{2}}$ **1/2**

$\therefore$ Particular solution is $x \cdot \sin\left(\frac{y}{x}\right) = \frac{1}{\sqrt{2}}$ **1/2**

32. Let N be the set of natural numbers and R be the relation on N × N defined by (a, b) R (c, d) iff ad = bc for all a, b, c, d ∈ N. Show that R is an equivalence relation.

Ans: Reflexive: For any (a, b) ∈ N × N

$a \cdot b = b \cdot a$

$\therefore$ (a, b) R (a, b) thus R is reflexive **1**

Symmetric: For (a, b), (c, d) ∈ N × N

(a, b) R (c, d) $\Rightarrow a \cdot d = b \cdot c$

$\Rightarrow c \cdot b = d \cdot a$

$\Rightarrow$ (c, d) R (a, b) $\therefore$ R is symmetric **$1\frac{1}{2}$**

Transitive : For any (a, b), (c, d), (e, f), ∈ N × N

(a, b) R (c, d) and (c, d) R (e, f)

$\Rightarrow$ $a \cdot d = b \cdot c$ and $c \cdot f = d \cdot e$

$\Rightarrow$ $a \cdot d \cdot c \cdot f = b \cdot c \cdot d \cdot e$ $\Rightarrow$ $a \cdot f = b \cdot e$

$\therefore$ (a, b) R (e, f), $\therefore$ R is transitive **$1\frac{1}{2}$**

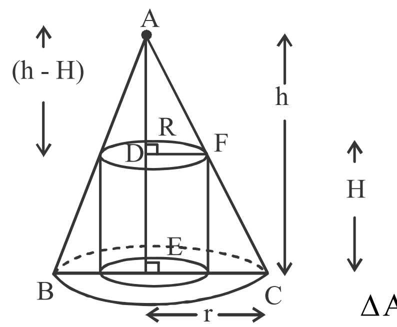
$\therefore$ R is an equivalance Relation

SECTION-D

Question numbers 33 to 36 carry 6 marks each.

33. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone.

Ans.



Let H = Height of cylinder

R = Radius of cylinder

Volume of cone $= \frac{\pi}{3} r^2 h$ **1/2**

V = Volume of cylinder $= \pi R^2 H$ **1/2**

$\Delta ADF \sim \Delta AEC \Rightarrow \frac{h-H}{h} = \frac{R}{r} \Rightarrow R = \frac{r}{R}(h-H)$ **1**

$\therefore V = \pi \cdot H \cdot \frac{r^2}{h^2}(h-H)^2 = \frac{\pi r^2}{h^2}\left(H^3 - 2hH^2 + Hh^2\right)$ **1**

$V'(H) = \frac{\pi r^2}{h^2}\left(3H^2 - 4hH + h^2\right), \ V'(h) = 0 \Rightarrow H = \frac{h}{3}$ **1+1**

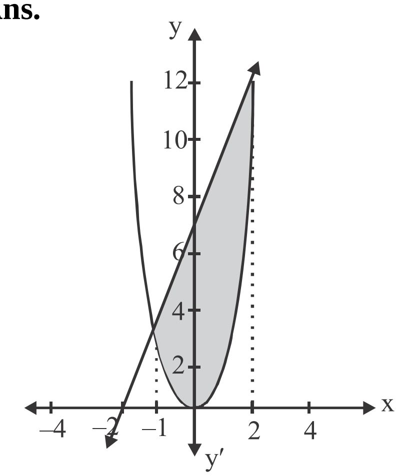
$V''(H) = \frac{\pi r^2}{h^2}(6H - 4h), \ V''\left(H = \frac{h}{3}\right) = \frac{\pi r^2}{h^2}(-2h) < 0$ **1/2**

$\therefore$ V is max iff $H = \frac{h}{3}$ and $R = \frac{2r}{3}$

$$\frac{\text{Volume of cylinder}}{\text{Volume of cone}} = \frac{3\pi R^2 H}{\pi r^2 h} = 3 \cdot \frac{4r^2}{9} \cdot \frac{h}{3} \cdot \frac{1}{r^2 h} = \frac{4}{9}$$ **1/2**

34. Using integration, find the area of the region enclosed by the parabola $y = 3x^2$ and the line $3x - y + 6 = 0$.

Ans.



Points of intersection x = –1, 2 **1**

Correct Graph **1**

Required area

$$= \int_{-1}^{2} 3(x+2)dx - 3\int_{-1}^{2} x^2 dx$$ **2**

$$= \frac{3}{2}\left[(x+2)^2\right]_{-1}^{2} - \left[x^3\right]_{-1}^{2}$$ **$1\frac{1}{2}$**

$$= \frac{3}{2} \times 15 - 9 = \frac{27}{2}$$ **1/2**

35. Find the equation of the plane that contains the point A(2,1,–1) and is perpendicular to the line of intersection of the planes $2x+y-z=3$ and $x+2y+z=2$. Also find the angle between the plane thus obtained and the y-axis.

Ans. Let equation of the required plane be:

$a(x-2)+b(y-1)+c(z+1)=0$ $\quad 1\frac{1}{2}$

Also : $2a+b-c=0$

$a+2b+c=0$

Solving: $\frac{a}{3}=\frac{b}{-3}=\frac{c}{3}=k \Rightarrow a=3k,\ b=-3k,\ c=3k$ $\quad 1\frac{1}{2}$

$\therefore$ Equation of plane is : $3k(x-2)-3k(y-1)+3k(z+1)=0$

$\Rightarrow x-y+z=0$ $\quad 1\frac{1}{2}$

Let angle between y-axis and plane = θ

then, $\sin\theta=\left|\frac{0-1+0}{\sqrt{1+1+1}}\right|=\left|\frac{-1}{\sqrt{3}}\right| \Rightarrow \theta=\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $\quad 1\frac{1}{2}$

OR

Find the distance of the point P(–2, –4, 7) from the point of intersection Q of the line $\vec{r}=(3\hat{i}-2\hat{j}+6\hat{k})+\lambda(2\hat{i}-\hat{j}+2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=6$. Also write the vector equation of the line PQ.

Ans. General point on line is: $\vec{r}=(3+2\lambda)\hat{i}+(-2-\lambda)\hat{j}+(6+2\lambda)\hat{k}$ $\quad 1$

For the point of intersection:

$\left[(3+2\lambda)\hat{i}+(-2-\lambda)\hat{j}+(6+2\lambda)\hat{k}\right]\cdot\left(\hat{i}-\hat{j}+\hat{k}\right)=6$ $\quad 1$

$\Rightarrow 3+2\lambda+2+\lambda+6+2\lambda=6 \Rightarrow \lambda=-1$ $\quad 1$

$\therefore Q(\hat{i}-\hat{j}+4\hat{k})=Q\,(1,-1,4)$ $\quad 1$

$PQ=3\sqrt{3}$, equation of the line PQ : $\vec{r}=-2\hat{i}-4\hat{j}+7\hat{k}+\mu\left(3\hat{i}+3\hat{j}-3\hat{k}\right)$ $\quad 1+1$

36. If $A=\begin{bmatrix}1 & 2 & -3\\ 3 & 2 & -2\\ 2 & -1 & 1\end{bmatrix}$, then find A^{-1} and use it to solve the following system of the equations:

$x+2y-3z=6$

$3x+2y-2z=3$

$2x-y+z=2$

Ans. $|A| = 7$; $\text{adj}(A) = \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}$; $A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{7}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}$ $1+1\frac{1}{2}+\frac{1}{2}$

The system of equations in Matrix form can be written as :

$A \cdot X = B$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$ **1**

$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ -35 \\ -35 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -5 \end{bmatrix}$ **1**

$\therefore$ $x = 1, y = -5, z = -5$ **1**

OR

Using properties of determinants, prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

Ans. $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$

$= \begin{vmatrix} b^2+c^2 & a^2 & bc \\ c^2+a^2 & b^2 & ca \\ a^2+b^2 & c^2 & ab \end{vmatrix}$ $(C_1 \to C_1 - 2C_3)$ **1**

$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix}$ $(C_1 \to C_1 + C_2)$ **1**

$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ 0 & b^2-a^2 & ca-bc \\ 0 & c^2-a^2 & ab-bc \end{vmatrix}$ $(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$ **2**

$= (b-a)(c-a)\begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ 0 & b+a & -c \\ 0 & c+a & -b \end{vmatrix}$ **1**

Expand along C_1

$$= \left(a^2 + b^2 + c^2\right)(b - a)\,(c - a)\left(-b^2 - ab + c^2 + ac\right)$$

$$= (a - b)\,(b - c)(c - a)(a + b + c)\left(a^2 + b^2 + c^2\right)$$ **1**