14. MATHEMATICAL REASONING

Statement: It is a sentence, which is either true or false, but not both simultaneously.

E.g.: a) India is in Asia. - a true sentence but a statement.

b) Kashmir is in Kerala - a false sentence but a statement.

Compound Statement: It is a statement formed by combining two or more simple statements. (a statement is said to be simple if it cannot be broken down into two or more sentences)

Truth value of a statement: The truth or falsity of a statement is called its truth value. If a statement is true, its truth value is True or T and if it is false, its truth value is False or F.

E.g.: The statement "Mumbai is the capital of Maharashtra" has truth value True (.T.).

Conjunction: If two simple statements p and q are connected by the word 'and', the resulting compound statement p and q is called a conjunction of p and q, and is written as $p \wedge q$

E.g.: p : Sona is in 5^{th} standard.

q : Mona is in 7^{th} standard.

$\Rightarrow p \wedge q$: Sona is in 5^{th} standard and Mona is in 7^{th} standard.

Disjunction: **I**f two simple statements p and q are connected by the word 'or', the resulting compound statement" p and q " is called disjunction of p and q is written as $"p \vee q"$.

E.g.: p : Ram is fat.

q : Shyam is tall.

$\Rightarrow p \vee q$: Ram is fat or Shyam is tall.

Negation: The denial of a statement is called the negation of the statement. The negation of a statement p is written as "$\sim p$ ".

E.g.: p : America is a super power.

$\sim q$: America is not a super power

Quantifiers: These are the phrases like, "there exist" and "for all".

E.g.: p : There exist a triangle whose all sides are equal.

q : For every prime p, $\sqrt{p}$ is irrational.

Contrapositive: The contrapositive of the statement, "if p, then q" is " if ~q, then ~p".

E.g.: " If a triangle is equilateral, then it is isosceles", is "If a triangle is not isosceles, then it is not equilateral".

Converse: The converse of a given statement "if p, then q" is " if q, then p".

E.g.: The converse of the statement "if a number n is even, then n^2 is even", is "if a number n^2 is even, then n is even"

Contradiction: This is a method to prove a statement to check whether a statement p is true, we assume that p is not true i.e., $\sim p$ is true. Then, we arrive at some result, which contradicts our assumption. Therefore, we conclude that p is true.

E.g.: Prove that $\sqrt{2}$ is irrational.

Let us assume that $\sqrt{2}$ is a rational number.

$\therefore$ $\sqrt{2}=\frac{a}{b}$, where a and b are co-prime. i.e., a and b have no common factors, which implies that

$2b^2=a^2 \Rightarrow 2$ divides a.

$\therefore$ there exists an integer 'k' such that $a=2k$

$\therefore$ $a^2=4k^2 \Rightarrow 2b^2=4k^2 \Rightarrow b^2=2k^2 \Rightarrow 2$ divides b.

i.e., 2 divides both a and b, which is contradiction to our assumption that a and b have no common factor.

$\therefore$ our supposition is wrong.

$\therefore$ $\sqrt{2}$ is an irrational number.

Note:

In the same manner, we can prove $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ $\sqrt{11}$, etc.

NCERT QUESTIONS

EXERCISE 14.1

1. Which of the following sentence are statements?
   a) There are 35 days in a month.
   b) Mathematics is difficult.
   c) The sum of 5 and 7 is greater than 10
   d) The square of a number is an even number.
   e) The side of a quadrilateral has equal length.
   f) Answer this question.
   g) The product of (–1) and 8 is 8
   h) The sum of all interior angles of a triangle is 180^0.
   i) Today is a windy day
   j) All real numbers are complex numbers.

Solution:

a) Since the maximum number of days in a month is 31. Therefore this sentence is a statement, which is false.

b) Mathematics is difficult is a true for some and false for others. Therefore this is not a statement.

c) 5+7=12>10, therefore, this is a true statement.

d) This is a false statement. Since $3^2 = 9$ is not even.

e) This is a sentence which can be true if all the sides are equal. But false otherwise. Therefore this is not a statement.

f) This not a statement.

g) Since $(-1)\times 8 = -8$. $\therefore$ This is a false statement.

h) This is a true statement. Since $\angle A + \angle B + \angle C = 180^0$, if A, B, C are the interior angles of a triangle.

i) Today is windy day is true on windy days and false for others. $\therefore$ it is not a statement.

j) Since $R \subseteq C$. Since all real numbers are complex numbers is a true statement. (A real number is a complex number with imaginary part zero)

2. Give three examples of sentence which are not statements. Give reasons for the answers.

i) Do your duty.

ii) How is your friend?

iii) How beautiful!

Solution:

iv) Do your duty.

It is not a statement. (Since it is an order)

v) How is your friend?

It is not a statement. (Since it is an interrogative type)

vi) How beautiful!

It is not a statement. (Since it is an exclamation)

EXERCISE 14.2

1. Write the negation of the following statements:

i) Chennai is capital of Tamil Nadu.

ii) $\sqrt{2}$ is not a complex number.

iii) All triangles are not equilateral .

iv) The number 2 is greater than 7.

v) Every natural number is an integer.

Solution:

i) p : Chennai is the capital of Tamil Nadu.

$\sim p$: Chennai is not the capital of Tamil nadu.

ii) p : $\sqrt{2}$ is not a complex number

$\sim p$: $\sqrt{2}$ is a complex number.

iii) p : All triangles are not equilateral triangles

$\sim p$: All triangles are equilateral triangles

iv) p : The number 2 is greater than seven

$\sim p$: The number 2 is not greater than seven

v) p : Every natural number is an integer

$\sim p$: Every natural number is not an integer.

2. Are the following pairs of statements negations of each other:

i) The number x is not a rational number

The number x is not an irrational number

Solution:

Let p : The number x is not a rational number

$\therefore$ $\sim p$: The number x is an irrational number

This is the same as the second statements. [Since if a number is not irrational, then it is rational

$\therefore$ the given pairs are negations of each other

ii) The number x is a rational number

The x is an irrational number

Solution:

Let p : The number x is a rational number

$\therefore$ $\sim p$: The number x is not a rational number

3. Find the component statements of the following compound statement and check whether they are true or false.

i) Number 3 is prime or it is odd.

ii) All integers are positive or negative.

iii) 100 is divisible by 3,11 and 5.

Solution:

i) The compound statements are:

p : number 3 is prime

q : It is odd

Both the component statements are true.

ii) The components statements are :

p : All integers are positive

q : All integers are negative

Here both the statements are false.

iii) The component statements are

p : 100 is divisible by 3

q : 100 is divisible bi 11

r : 100 is divisible by 5

Here p,q are false and r is true.

EXERCISE 14.3

1. For each of the following compound statements first identify the connecting words and then break it into component statements:

   i) All rational numbers are real and all real numbers are not complex.

   ii) Square of an integer is positive or negative.

   iii) The sand heats up quickly in the sun and dues not cool down fast at night.

   iv) $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.

Solution:

i) Here the connecting words is 'and'

The components statements are:

p : All rational numbers are real.

q : All real numbers are not complex.

ii) Here the connecting word is 'or'

The component statements are:

p : square of an integer is positive.

q : square of an integer is negative.

iii) Here the connecting word is 'and'

The component statements are

p : The sand heats up quickly in the sun.

q : The sand does not cool down fast at night.

iv) Here the connecting word is 'and'

The component statements are:

p : $x = 2$ is a root of the equation $3x^2 - x - 10 = 0$

q : $x = 3$ is a root of the equation $3x^2 - x - 10 = 0$.

2. Identify quantifier in the following statements and write the negation of the stat5ements.

i) There exist number which is equal to its square.

ii) For every real number x , x is less than $x+1$.

iii) There exists a capital for every state in India

Solution:

i) Quantifier is "There exists"

Negation: There does not exist a number which is equal to its square.

ii) Quantifier is " For every"

Negation: For every real number x , x is not less than $x+1$.

iii) Quantifier " There exist"

Negation: There does not exist a capital for every state in India.

3. Check whether the following pair of statements are negations of each other. Given reason for your answer

i) $x + y = y + x$ is true for every real numbers x and y

ii) There exist a real numbers x and y for which $x + y = y + x$

Solution: (i) and (ii) are not negations of each other.

Negation of (i): $x + y = y + x$ is not true for every real numbers x and y .

Negation of (ii): There does not exist real numbers x and y for which $x + y = y + x$.

4. State whether the "Or" used in the following statements is "exclusive" or inclusive for your answer.

i) Sun rises or Moon sets.

ii) To apply for a driving license, you should have a ration card or a passport.

iii) All integers are positive or negative.

Solution:

i) Here 'Or' used is exclusive because "Sun rises" and "Moon sets" cannot be true simultaneously.

ii) Here 'Or' used is inclusive because you can apply for driving license when you have both ration card and passport.

iii) Here 'Or' used in exclusive because an integer cannot be both positive as well as negative.

EXERCISE 14.4

1. Rewrite the following statement with "if- then " in five different ways conveying the same

   If a natural number is odd, then its square is also odd.

Solution:

i) A natural number is odd implies its square is also odd.

ii) A natural number is odd only if its square is odd.

iii) When a natural number is odd, then its square is necessarily odd.

iv) A natural number is odd is sufficient to conclude that its square is also odd.

v) If square of a natural number is not odd, then the natural number is not odd.

2. Write the contrapositive and converse of the following statements:

   i) If x is a prime number, then x is odd.

   ii) If the two lines are parallel, then they do not intersect in the same plane.

   iii) Something is cold implies that it has low temperature.

   iv) You cannot comprehend geometry if you do not know how to reason deductively.

   v) x is an even number implies that x is divisible by 4.

Solution:

i) Here p : x is a prime number.

q : x is odd.

$\sim p$: x is not a prime number.

$\sim q$: x is not odd.

Contra positive of a given statement is:

If x is not odd, then x is not a prime number.

Converse of given statement is

If x is odd, then x is a prime number.

ii) Let p : Two lines are parallel.

q : They do not intersect in the same plane.

$\sim p$: Two lines are not parallel.

$\sim q$: They intersect in the same plane.

Contra positive statement is:

If two lines intersect in the same plane then they are not parallel.

Converse of given statement is:

If two lines do not intersect in the same plane they are parallel.

iii) p : Something is cold.

q : It has low temperature.

$\sim p$: Something is not cold.

$\sim q$: It has not low temperature.

Contra positive statement is:

If something does not have low temperature then it is not cold.

Converse of given statement is:

If something has low temperature then it is cold.

iv) p : You cannot comprehend geometry.

q : You do not know how to reason deductively.

$\sim p$: You can comprehend geometry.

$\sim q$: You know how to reason deductively.

Contra positive statement is:

If you know how to reason deductively, then you can comprehend geometry.

Converse statement is:

If you do not know how to reason deductively, you cannot comprehend geometry.

v) p: is an even number.

q : is divisible by 4

~p: is not an even number.

~q: is not divisible by 4.

Contra positive statement is:

If x is not divisible by 4 then x is not an even number .

3. Write each of the following statements in the form "if-then"

i) You get a job implies that your credentials are good.

ii) The Banana tree will bloom if it says warm for month.

iii) A quadrilateral is a parallelogram if its diagonals bisect each other.

iv) To get an A+ in the class, it is necessary that you do all the exercises of the book.

Solution:

i) If you get a job then your credentials are good

ii) If it stays warm for a month then the banana trees will bloom.

iii) If the diagonals of a quadrilateral bisect each other then it is a parallelogram.

iv) If you get A+ in the class then you have done all the exercises of the book.

4. Given statements in (a) and (b), identify the statements given below as contrapositive or converse each other.

   a) If you live in Delhi, then you have winter clothes.

      i) If you do not have winter clothes, then you do not live in Delhi.

      ii) If you have winter clothes, then you live in Delhi.

   b) If a quadrilaterals a parallelogram, then its diagonals bisect each other.

      i) If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.

      ii) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

Solution.

a) (i) contra positive statement.

   (ii) converse statement.

b) (i) contra positive statement.

   (ii) converse statement.

EXERCISE 14.5

1. Show that the statement

   p : " If x is a real number such that $x^3+4x=0$, then x is 0" is true by

   (i) direct method,

   (ii) method of contradiction,

   (iii) method of contra positive.

Solution:

i) Direct method:

$$x^3+4x,=0 \Rightarrow x\left(x^2+4\right)=0 \Rightarrow x=0$$

ii) Contradiction method:

If possible , suppose $x \neq 0 \Rightarrow x^2 \neq 0 \Rightarrow x^2+4 \neq 0$

$\Rightarrow x\left(x^2+4\right) \neq 0 \Rightarrow x^3+4x \neq 0$, a contradiction

$\therefore$ our supposition is wrong.

$\therefore \; x=0$.

iii) Contra positive method:

Let $p: x^3+4x=0 \;\; and \;\; q: x=0$

$\therefore \; \sim p: x^3+4x \neq 0 \quad and \quad \sim q: x \neq 0$

$Now \;\; x \neq 0 \Rightarrow x^2+4 \neq 0 \Rightarrow x\left(x^2+4\right) \neq 0 \Rightarrow x^3+4x \neq 0$

$\therefore \sim q \Rightarrow \sim p$

$\therefore \; p \Rightarrow q$

$i.e. \;\; x^3+4x=0$

$\Rightarrow x=0.$

2. Show that the statement "For any real number a and b, $a^2=b^2$ implies that a=b" giving a counter- example.

Solution:

Let $a=-3 \;\; and \;\; b=3$

Now a^2-b^2 but $a \neq b$. $\therefore$ the given statement is not true.

3. Show that the following statement is true by the method of contra positive.

p: if x is an integer and x^2 is even, then x is also even.

Solution:

Let p: if x is an integer and x^2 is even

q: x is even.

Let q be false therefore $\sim q$ is true . $\therefore$ x is odd integer.

Let $x = 2m+1 \Rightarrow x^2 = (2m+1)^2$

$\Rightarrow x^2 = 4m^2 + 4m + 1 = 2\left(2m^2 + 2m\right) + 1 = 2k + 1,$ odd integer, where $k = 2m^2 + 2m$.

Therefore p *is false. Thus* $\sim q \Rightarrow \sim q$. Hence the given statement is true.

4. By giving a counter example, show that following statements are not true.

i) p : If all the angles of a triangle are equal, then the triangle is an obtuse angled triangle.

ii) q : The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2.

Solution:

i) Consider an equilateral triangle ABC. , then $\angle A = \angle B = \angle C = 60^0$. Thus all the three angles of the triangle are equal to 60^0 (each).

$\therefore$ triangle is not an obtuse angled triangle. Hence the given statements is not true.

ii) Since the equation $x^2 - 1 = 0 \Rightarrow x^2 = 1 \Rightarrow 1 \Rightarrow x = \pm 1$ and 1 lies between 0 and 2. Therefore given statement is not true.

5. Which of the following statements are true and which are false? In each case give a valid reason for saying so.

i) p : Each radius of a circle is a chord of the circle.

ii) q : The centre of a circle bisects each chord of the circle.

iii) r : Circle is a particular case of an ellipse.

iv) s : if x and y are integers such that $x > y$, then $-x < -y$.

v) t : $\sqrt{11}$ is a rational number.

Solution:

i) p is false. Since radius of the circle meets the circle in only one point, whereas chord of the circle is a straight line meeting the circle in two points.

ii) The centre of a circle bisects each of the circle is false, since if a chord does not pass through the centre, then it is not bisected by the centre.

iii) Circle is a particular case of an ellipse. This is true when major axis = minor axis

iv) This is a true statement $x > y \Rightarrow -x < -y \; \forall$ integers x, y.

v) Let us assume that $\sqrt{11}$ is a rational number.

Let us assume that $\sqrt{11}$ is a rational number.

$\therefore \; \sqrt{11} = \frac{a}{b}$, where a and b are co-prime. i.e., a and b have no common factors, which implies that $11b^2 = a^2 \Rightarrow 11$ divides a.

$\therefore$ there exists an integer 'k' such that $a = 11k$

$\therefore \; a^2 = 121k^2 \Rightarrow 11b^2 = 121k^2 \Rightarrow b^2 = 11k^2 \Rightarrow 11$ divides b.

i.e., 11 divides both a and b, which is contradiction to our assumption that a and b have no common factor.

$\therefore$ our supposition is wrong.

$\therefore \; \sqrt{11}$ is an irrational number.

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