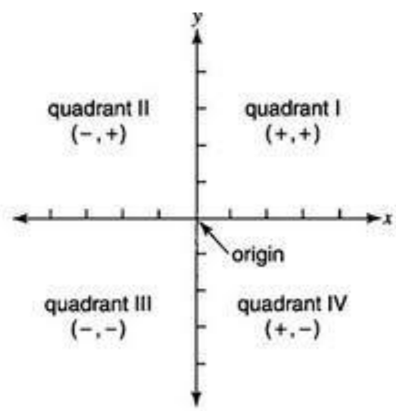
10. STRAIGHT LINES

CO-ORDINATES

1. Rectangular coordinate system

A system obtained by taking two mutually $\perp r$ lines in a plane. The horizontal line is known as x-axis and the vertical line is known as y-axis. The point of intersection of the two axes is known as origin, denoted by O and its coordinates is (0,0).

The two mutually perpendicular lines divide a plane into four regions, each region is known as quadrant. Quadrants are taken in the anti-clockwise direction from the positive x-axis.



quadrant in graph

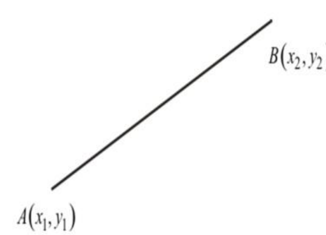
2. Signs of co-ordinates in different quadrants

Quadrants / Co-ordinates	I	II	III	IV
x	+	–	–	+
y	+	+	–	–

3. Distance formula.

Distance between two points A(x_1, y_1) and B $(x_2, y_2)$is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (or) \quad AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



E.g.: a) Find the distance between two points A(2,3) and B (5, -1).

$$AB = \sqrt{(5-2)^2 + (-1-3)^2}$$
$$= \sqrt{3^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5 units$$

b) $A(a\cos\theta, a\sin\theta)$ *and* $B(b\cos\theta, b\sin\theta)$

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(b\cos\theta - a\cos\theta)^2 + (b\sin\theta - a\sin\theta)^2} \\ &= \sqrt{[\cos\theta(b-a)]^2 + [\sin\theta(b-a)]^2} \\ &= \sqrt{(b-a)^2\left[\cos^2\theta + \sin^2\theta\right]} = \sqrt{(b-a)^2 \times 1} = (b-a) \end{aligned}$$

4. Distance from origin to a point:

E.g.: Find the distance from origin to a point (3,4)

$$OP = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5\,units$$

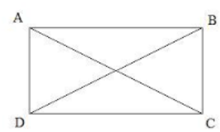
5. Using distance formula, we can prove that the given points are the vertices of a:

a) rectangle

AB = CD

AB = BC

AC = BD

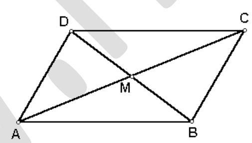


b) Parallelogram

AB = DC

AD = BC

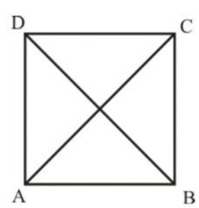
AC $\neq$ BD



c) Square

AD=AB=BC=CD

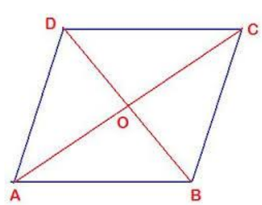
AC = BD



d) Rhombus

AB=DC = AD = BC

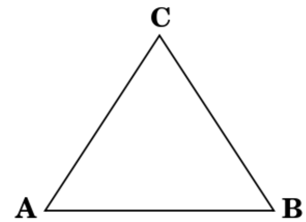
AC $\neq$ BD



e) Isosceles triangle

Any two sides are equal.

AB = AC (or) BC = BA (or) CB = CA



f) Equilateral triangle

AB = BC = AC

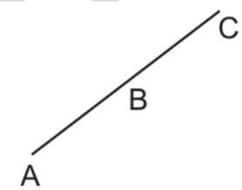
g) Right angled triangle

Using 'Pythagoras' theorem, square of the largest side is equal to sum of the squares of other two sides.

$$i.e.,\ AB^2 = BC^2 + AC^2 \quad (or)\ BC^2 = AB^2 + AC^2 \quad (or)\ AC^2 = AB^2 + BC^2$$

h) Right angled isosceles triangle

(e) + (g)

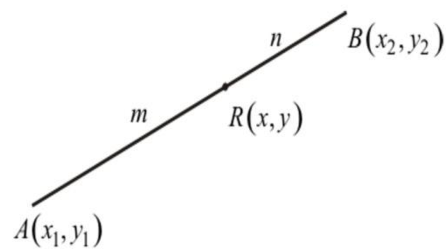


i) Collinear points: If three points lie on a line, then they are known as collinear points.

If three points A,B,C are collinear, then AB + BC = AC (or) BC +AC = AB

(or) AC + AB = BC.

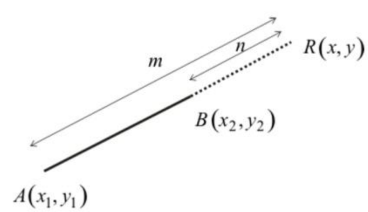
6. Section formula



Coordinates of a point R which divides the line joining two points $A(x_1 y_1)$ and $B(x_2, y_2)$ in the ratio $m:n$

a) internally is

$$R = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$



b) externally is

$$R = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - my_1}{m-n}\right)$$

7. Midpoint formula

If R (x,y) be the midpoint of AB then the ratios m and n are equal.

$$\therefore R = \left(\frac{mx_2 + mx_1}{m+m}, \frac{my_2 + my_1}{m+m}\right)$$

$$= \left(\frac{m(x_2 + x_1)}{2m} \frac{m(y_2 + y_1)}{2m}\right)$$

$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$ is known as midpoint formula.

Note: Using midpoint formula, we can prove the points are the vertices of a rectangle, parallelogram, square and rhombus using the formula,

midpoint of diagonal AC = midpoint of diagonal BD

8. Area of a triangle.
   If A, B and C be the vertices of a triangle ABC, then

   Area of a $\Delta ABC = \frac{1}{2}\left[x_1\left(y_2-y_3\right)+x_2\left(y_3-y_1\right)+x_3\left(y_1-y_2\right)\right]$

(or)

Area of triangle ABC= $\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$

$$=\frac{1}{2}\left[x_1y_2+x_2y_3+x_3y_1-x_2y_1-x_3y_2-x_1y_3\right]$$

E.g.: Find the the area of the triangle with vertices A(2,1) B (3,5) and C(0, -2)

$$\text{Ar}\left(\Delta ABC\right)=\frac{1}{2}\left[x_1\left(y_2-y_3\right)+x_2\left(y_3-y_1\right)+x_3\left(y_1-y_2\right)\right]$$

$$=\frac{1}{2}\left[2\left(5+2\right)+3\left(-2-1\right)+0\left(1-5\right)\right]$$

$$=\frac{1}{2}\left[2\left(7\right)+3\left(-3\right)\right]$$

$$=\frac{1}{2}\left[14-9\right]$$

$$=\frac{1}{2}\left(5\right)=\frac{5}{2}\,sq\,units$$

Alternate Method:

$$\text{Ar }\left(\Delta ABC\right)=\frac{1}{2}\begin{vmatrix} 2 & 1 \\ 3 & 5 \\ 0 & -2 \\ 2 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left[10+-6+0-3-0--4\right]=\frac{1}{2}\left[10-6-3+4\right]$$

$$=\frac{1}{2}\left[14-9\right]=\frac{5}{2}\,sq\,units$$

9. Collinearity of 3 points: If A,B,C are collinear, then $\boxed{\text{ar}\left(\Delta ABC\right)=0}$

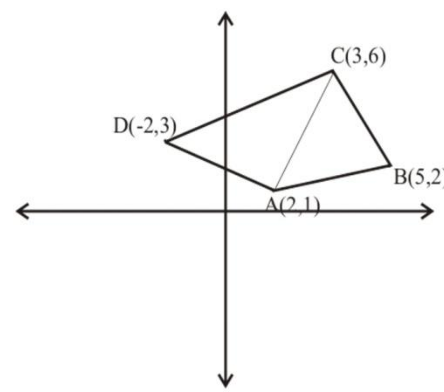
10. If A, B, C and D are the vertices of a quadrilateral ABCD, then

Area of quadrilateral ABCD, A=ar(ΔADC) + ar(ΔABC)

(OR)

$$A=\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ y_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_1 & y_1 \end{vmatrix}$$

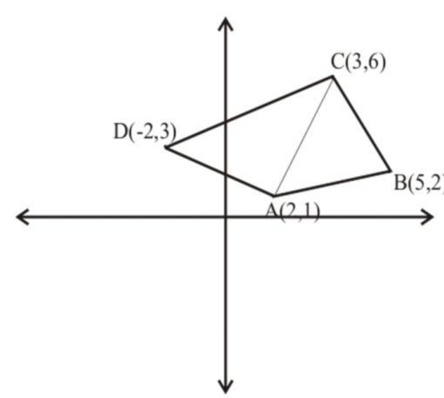
Find the area of the quadrilateral ABCD, having vertices A(2,1), B(5,2), C(3,6) and D(-2,3).



Area of $\Delta ADC = \frac{1}{2}\left[2(6-3)+3(3-1)+-2(1-6)\right]$

$$=\frac{1}{2}\left[2(3)+3(2)-2(-5)\right]=\frac{1}{2}[6+6+10]$$
$$=\frac{1}{2}(22)=11\,sq\,units$$

Area of $\Delta ABC = \frac{1}{2}\left[2(2-6)+5(6-1)+3(1-2)\right]$



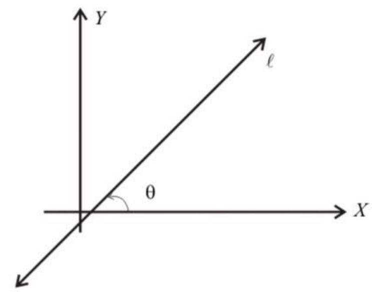
$$=\frac{1}{2}\left[2(-4)+5(5)+3(-1)\right]=\frac{1}{2}[-8+25-3]$$
$$=\frac{1}{2}(14)=7\,sq.units$$

$\therefore$ area of the quadrilateral ABCD $=11+7=18\,sq.units$

Alternate Method:

$$A=\frac{1}{2}\begin{vmatrix} 2 & 1 \\ 5 & 2 \\ 3 & 6 \\ -2 & 3 \\ 2 & 1 \end{vmatrix}=\frac{1}{2}\left[4+30+9+(-2)-5-6-(-12)-6\right]$$
$$=\frac{1}{2}[4+30+9-2-5-6+12-6]$$
$$=\frac{1}{2}[4+30+9-2-5]=\frac{1}{2}(36)=18\,sq.units$$

11. SLOPE OF A LINE

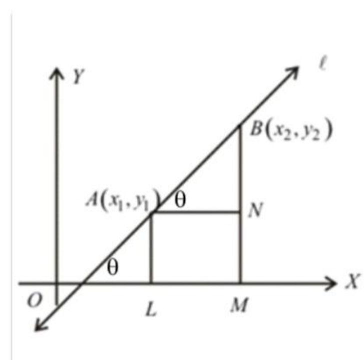


- If the inclination of a line with the +ve direction of the x-axis is θ, then slope of the line, $\boxed{m = \tan\theta}$

1	0^0	$m = \tan 0 = 0 \quad (\text{the line is} \parallel \text{to } x-\text{axis or} \perp r \text{ to } y-\text{axis})$
2	30^0	$m = \tan 30 = \frac{1}{\sqrt{3}}$
3	45^0	$m = \tan 45 = 1$
4	60^0	$m = \tan 60^0 = \sqrt{3}$
5	90^0	$m = \tan 90 = \infty \quad (a \text{ line is} \perp r \text{ to } x-\text{axis or} \parallel el \text{ to } y-\text{axis})$
6	120^0	$m = \tan 120 = \tan(180-60) = -\tan 60 = -\sqrt{3}$
7	150^0	$m = \tan 150 = \tan(180-30) = -\tan 30 = -\frac{1}{\sqrt{3}}$

- Slope of the line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad (or) \quad m = \frac{y_1 - y_2}{x_1 - x_2}$$

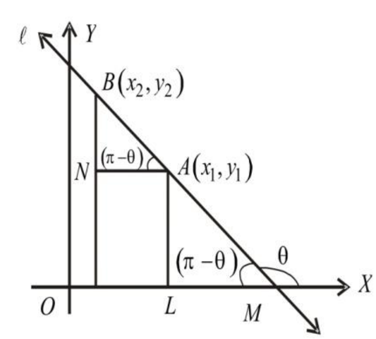


(Case i) : When θ is acute $\left(<90^0\right)$.

$$\angle BAN = \angle BTX = \theta$$
$$AN = LM = OM - OL = x_2 - x_1$$
$$BN = BM - NM = BM - AL = y_2 - y_1$$

In $\Delta BAN, \tan\theta = \frac{BN}{AN} \Rightarrow \quad m = \frac{y_2 - y_1}{x_2 - x_1}$ .... (1)

Case ii) when θ is obtuse $\left(>90^0\right)$.



$$\angle BAN = \angle BTO = 180 - \theta$$
$$NA = ML = OL - OM = x_1 - x_2$$
$$BN = BM - NM = BM - AL = y_2 - y_1$$

In ΔBAN, $\tan(180-\theta)=\frac{BN}{NA}$

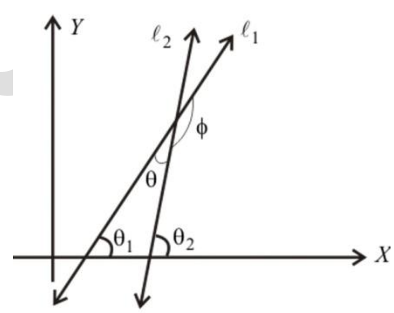
$$-\tan\theta=\frac{y_2-y_1}{x_1-x_2}$$

$$-\tan\theta=\frac{y_2-y_1}{-(x_2-x_1)}$$

$$\tan\theta=\frac{y_2-y_1}{x_2-x_1}\Rightarrow m=\frac{y_2-y_1}{x_2-x_1} \quad ....(2)$$

From (1) and (2), we have slope of a line, $\boxed{m=\frac{y_2-y_1}{x_2-x_1}}$

- Angle between the lines having slopes m_1 and m_2 is $\tan\theta=\frac{m_2-m_1}{1+m_1m_2}$.
- Acute angle between the lines having slopes m_1 and m_2 is $\tan\theta=\left|\frac{m_2-m_1}{1+m_1m_2}\right|$



- If three points A, B and C are collinear or lie on a line, then
  Slope of AB = slope of BC (or) Slope of AB = slope of AC (or) Slope of AC = slope of BC
- If two lines are parallel, then slopes are equal. i.e., $m_1=m_2$
- If two lines are perpendicular, then product of their slopes is equal to -1.
  i.e., $m_1m_2=-1$.

Equation of a straight line

- Equation of the x axis is $y=0$
- Equation of the y axis is $x=0$
- Equation of a straight line parallel to the x axis is $y=b$
- Equation of a straight line parallel to the axis is $x=a$
- Equation of a straight line having slope 'm' and y intercept 'c' is $y=mx+c$
- Equation of a straight line having slope 'm' and passing through a point (x_1,y_1) is $y-y_1=m(x-x_1)$
- Equation of a straight line passing through two points (x_1,y_1) and (x_2,y_2) is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$
- Equation of a straight line making intercepts 'a' and 'b' on the coordinate axes is $\frac{x}{a}+\frac{y}{b}=1$
- Equation of a straight line whose perpendicular distance from the origin is 'p' and the perpendicular makes an angle ω with the positive direction of the x axis is $x\cos\omega+y\sin\omega=p$

- Equation of a straight line parallel to a given line $Ax+By+C=0$ is $Ax+By+K=0$, where 'K' is any constant.
- Equation of a straight line perpendicular to a given line $Ax+By+C=0$ is $Bx-Ay+K=0$, where 'K' is any constant.
- General form of a straight line is $Ax+By+C=0$, where A, B and C are constants.
- Reduction into slope-intercept form:

  General form of a straight line is $Ax+By+C=0$

  $$By=-Ax-C$$
  $$y=\frac{-Ax-C}{B}$$
  $$y=\left(-\frac{A}{B}\right)x+\left(-\frac{C}{B}\right)$$

  Comparing with $y=mx+c$, we have,

  i. $slope,\ m=-\frac{A}{B}$

  ii. y-intercept, $c=-\frac{C}{B}$

- Reduction into intercept form:

  General form of a straight line is $Ax+By+C=0$

  $$Ax+By=-C$$
  $$\frac{Ax}{-C}+\frac{By}{-C}=1 \qquad (dividing\ by\ -C)$$
  $$\frac{x}{\left(-\frac{C}{A}\right)}+\frac{y}{\left(-\frac{C}{B}\right)}=1$$

  Comparing with $\frac{x}{a}+\frac{y}{b}=1$, we have,

  i. x-intercept, $a=-\frac{C}{A}$

  ii. y-intercept, $b=-\frac{C}{B}$

- Reduction into normal form:

  Let $x\cos\omega+y\sin\omega=p$ be the normal form of the equation of a straight line is $Ax+By+C=0$ or $Ax+By=-C$, then $\left(\pm\frac{A}{\sqrt{A^2+B^2}}\right)x+\left(\pm\frac{B}{\sqrt{A^2+B^2}}\right)y=\pm\frac{C}{\sqrt{A^2+B^2}}$ is the normal form of the straight line.

  Note: The perpendicular distance from origin the line $Ax+By+C=0$ is $p=\left|\frac{C}{\sqrt{A^2+B^2}}\right|$.

E.g.: Reduce the equation $\sqrt{3}x+y-8=0$ into normal form. Also find p and ω.

$$\sqrt{3}x+y=8 \ ....\ (1)$$

$$\sqrt{A^2+B^2}=\sqrt{\left(\sqrt{3}\right)^2+1^2}=\sqrt{3+1}=\sqrt{4}=2$$

$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y=\frac{8}{2}\Rightarrow\frac{\sqrt{3}}{2}x+\frac{1}{2}y=4$$

Comparing with $x\cos\omega+y\sin\omega=p$

x *and* y *in I quadrant.*

$$\cos\omega=\frac{\sqrt{3}}{2}\Rightarrow\omega=30^\circ$$

$\therefore x\cos 30^\circ+y\sin 30^\circ=4$, is the normal form.

Here, $\omega=30^\circ$, $p=4$.

- Perpendicular distance from one point (x_1,y_1) to a line $Ax+By+C=0$ is

  $$d=\left|\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right|.$$

- Distance between parallel lines $Ax_1+By_1+C_1=0$ and $Ax_2+By_2+C_2=0$ is $d=\left|\frac{C_2-C_1}{\sqrt{A^2+B^2}}\right|$.

- Point of intersection between the lines $Ax_1+By_1+C_1=0$ and $Ax_2+By_2+C_2=0$.

  Either solving the two lines (using the solution of simultaneous linear equations in 2 unknowns) or using the formula $(x,y)=\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$.

- Concurrent lines: If three or more lines are intersecting at a point, then the lines are known as concurrent lines.

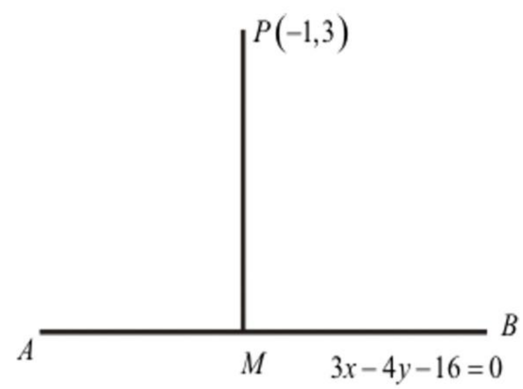
- To prove that the given three lines are concurrent:

  i. Find the point of intersection of any two lines,

  ii. Substitute this point in the third line,

  iii. If it satisfies, then the lines are concurrent, otherwise not concurrent.

- To find the foot of the perpendicular drawn from one point to a line.



Equation of AB is $3x-4y-16=0$ ....(1)

Slope of AB $=-\frac{A}{B}=-\frac{3}{-4}=\frac{3}{4}$

$\therefore$ Slope of PM $=\frac{-1}{slope\ of\ AB}=-\frac{4}{3}$ $\quad[\because AB\perp PM]$

Equation of PM: $y-y_1=m(x-x_1)$

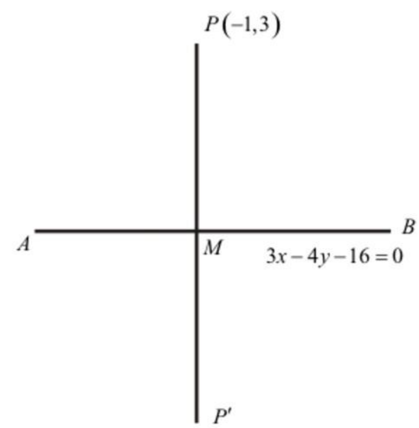
$$y-3=-\frac{4}{3}(x--1)$$
$$3y-9=-4(x+1)$$
$$3y-9=-4x-4 \Rightarrow 3y-9+4x+4=0 \Rightarrow 4x+3y-5=0 \text{ ....(2)}$$

Solving (1) and (2), we have

$$(1)\times 3+(2)\times 4 \Rightarrow$$
$$9x-12y-48=0$$
$$16x+12y-20=0$$
....
$$25x \qquad -68=0$$
$$\therefore 25x=68 \Rightarrow x=\frac{68}{25}$$
$$in\ (2)$$
$$4\times\frac{68}{25}+3y-5=0$$
$$3y=5-\frac{272}{25} \Rightarrow 3y=\frac{125-272}{25} \Rightarrow 3y=\frac{-147}{25}$$
$$\Rightarrow y=\frac{-49}{25}$$

$\therefore$ the foot of the perpendicular from $P(-1,3)$ to the line $3x-4y-16=0$ is $M\left(\frac{68}{25},-\frac{49}{25}\right)$.

- To find the image of the point to a line.



Equation of AB is $3x-4y-16=0$ ....(1)

Slope of AB $=-\frac{A}{B}=-\frac{3}{-4}=\frac{3}{4}$

$\therefore$ Slope of PM $=\frac{-1}{slope\ of\ AB}=-\frac{4}{3}$ $[\because AB\perp PM]$

Equation of PM: $y-y_1=m(x-x_1)$

$$y-3=-\frac{4}{3}(x--1)$$
$$3y-9=-4(x+1)$$
$$3y-9=-4x-4 \Rightarrow 3y-9+4x+4=0 \Rightarrow 4x+3y-5=0 \text{ ....(2)}$$

Solving (1) and (2), we have

$(1)\times 3+(2)\times 4 \Rightarrow$
$9x-12y-48=0$
$16x+12y-20=0$
....
$25x \qquad -68=0$

$\therefore 25x=68 \Rightarrow x=\dfrac{68}{25}$

$in\ (2)$

$4\times\dfrac{68}{25}+3y-5=0$

$3y=5-\dfrac{272}{25} \Rightarrow 3y=\dfrac{125-272}{25} \Rightarrow 3y=\dfrac{-147}{25}$

$\Rightarrow y=\dfrac{-49}{25}$

$\therefore$ the foot of the perpendicular from $P(-1,3)$ to the line $3x-4y-16=0$ is $M\left(\dfrac{68}{25},-\dfrac{49}{25}\right)$.

Now M is the midpoint of PP', using midpoint formula,

$\dfrac{x+-1}{2}=\dfrac{68}{25} \Rightarrow x-1=\dfrac{136}{25} \Rightarrow x=\dfrac{136}{25}+1=\dfrac{136+25}{25}=\dfrac{161}{25}$

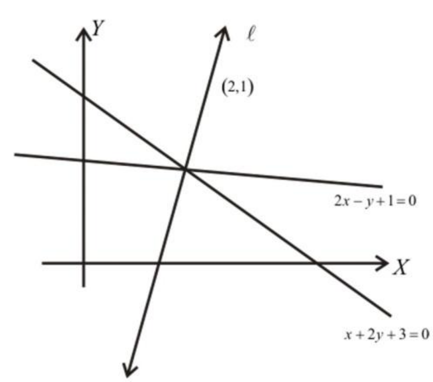
$\dfrac{x+3}{2}=-\dfrac{49}{25} \Rightarrow x+3=-\dfrac{98}{25} \Rightarrow x=-\dfrac{98}{25}+1=\dfrac{-98+25}{25}=-\dfrac{73}{25}$

$\therefore$ the image is $\left(\dfrac{161}{25},-\dfrac{73}{25}\right)$.

- Equation of a straight line passing through the point of intersection of the lines $L_1: Ax_1+By_1+C_1=0$ and $L_2: Ax_2+By_2+C_2=0$ is $L_1+kL_2=0$, where 'k' be any constant.

E.g.:

1. Find the equation of a straight line passing through the point of intersection of the lines $2x-y+1=0$ and $x+2y+3=0$ passing through (2,1).



Required equation is $L_1+kL_2=0$

$2x-y+1+k(x+2y+3)=0$ .... (1)

Since (1) passes through (2,1)

$2(2)-(1)+4+k\left[(2)+2(1)+3\right]=0$

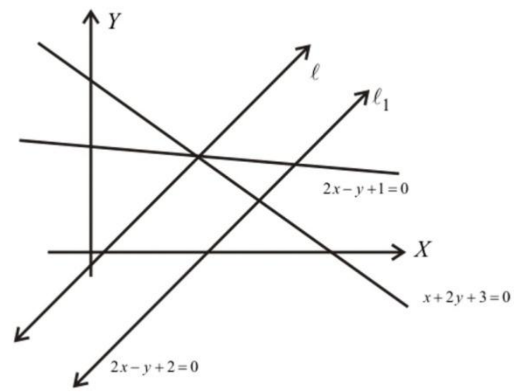
$4-1+4+k(2+2+3)=0$

$7+7k=0 \Rightarrow 7k=-7 \Rightarrow k=-1$

In (1), we have, $2x-y+1+-1(x+2y+3)=0$

$2x-y+1+-x-2y-3=0 \Rightarrow x-3y-2=0$

2. Find the equation of a straight line passing through the point of intersection of the lines $2x-y+1=0$ and $x+2y+3=0$ parallel to the line $2x-y+2=0$.



Required equation is $L_1+kL_2=0$

$2x-y+1+k(x+2y+3)=0$ .... (1)

$(2+k)x+(2k-1)y+(3k+1)=0$ .... (2)

Slope of (2) is $m_1=-\frac{A}{B}=-\frac{2+k}{2k-1}$

Slope of the given line is $m_2=-\frac{A}{B}=-\frac{2}{-1}=2$

Since the lines are parallel, slopes are equal. i.e.,

$m_1=m_2\Rightarrow -\frac{2+k}{2k-1}=2$

$\Rightarrow -(2+k)=2(2k-1)$

$\Rightarrow -2-k=4k-2\Rightarrow -2+2=4k+k$

$\Rightarrow 5k=0\Rightarrow k=0$

Since (1) passes through (2,1)

$2(2)-(1)+4+k\left[(2)+2(1)+3\right]=0$

$4-1+4+k(2+2+3)=0$

$7+7k=0\Rightarrow 7k=-7\Rightarrow k=-1$

In (1), we have, $2x-y+1+-1(x+2y+3)=0$

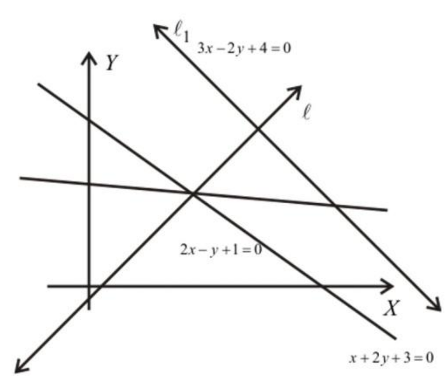
$2x-y+1+-x-2y-3=0\Rightarrow x-3y-2=0$

In (1), we have,

$2x-y+1+0(x+2y+3)=0$

$2x-y+1=0$ is the required equation.

3. Find the equation of a straight line passing through the point of intersection of the lines $2x-y+1=0$ and $x+2y+3=0$ perpendicular to the line $3x-2y+4=0$.



Required equation is $L_1+kL_2=0$

$2x-y+1+k(x+2y+3)=0$ .... (1)

$(2+k)x+(2k-1)y+(3k+1)=0$ .... (2)

Slope of (2) is $m_1=-\frac{A}{B}=-\frac{2+k}{2k-1}$

Slope of the given line is $m_2=-\frac{A}{B}=-\frac{-3}{-2}=-\frac{3}{2}$

Since the lines are perpendicular,

$m_1m_2=-1\Rightarrow -\frac{2+k}{2k-1}\times -\frac{3}{2}=1$

$$\frac{2+k}{2k-1}\times\frac{3}{2}=1 \Rightarrow \frac{3(2+k)}{2(2k-1)}=1 \Rightarrow \frac{6+3k}{4k-2}=1$$

$6+3k=4k-2 \Rightarrow 6+2=4k-3k \Rightarrow k=8$

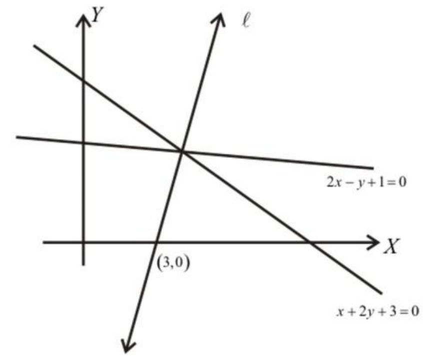
In (1), we have,

$2x-y+1+8(x+2y+3)=0$

$2x-y+1+8x+16y+24=0 \Rightarrow 10x+15y+25=0$

$2x+3y+5=0$ is the required equation.

4. Find the equation of a straight line passing through the point of intersection of the lines $2x-y+1=0$ and $x+2y+3=0$ and has x intercept 3.



Required equation is $L_1+kL_2=0$

$2x-y+1+k(x+2y+3)=0$ ………… (1)

Since the required line has x intercept 3, (1) passes through the point (3,0)

$2(3)-0+1+k(3+2(0)+3)=0$

$6+1+k(6)=0 \Rightarrow 6k=-7 \Rightarrow k=-\frac{7}{6}$

In (1), we have,

$2x-y+1+-\frac{7}{6}(x+2y+3)=0$

$12x-6y+6-7(x+2y+3)=0$

$12x-6y+6-7x-14y-21=0$

$5x-20y-15=0$, is the required equation.