4. PRINCIPLE OF MATHEMATICAL INDUCTION

The principle of mathematical induction is a mathematical process which is used to establish the validity of a general result involving natural numbers.

The Principle of Mathematical Induction

If P(n) be a mathematical statement ($n \in N$) , such that

i. P(1) is true
ii. Assume that P(k) be true,
iii. To prove that P(k+1) is true, then P(n) is true for all positive integral values of n.

Using principle of mathematical induction, prove the following:

1. $1+2+3+...+n=\frac{n(n+1)}{2},\ n\in N$

$$P(n):1+2+3+...+n=\frac{n(n+1)}{2}$$

$$Let\ \ P(1):1=\frac{1(1+1)}{2}\Rightarrow 1=1$$

Hence, $P(1)$ *is true.*

Assume that $P(k)$ *be true.*

$$P(k):1+2+3+...+k=\frac{k(k+1)}{2}$$

To prove that $P(k+1)$ *is true.*

$$P(k+1):1+2+3+...+k+(k+1)=\frac{(k+1)(k+2)}{2}$$

$$\Rightarrow P(k)+(k+1)=\frac{(k+1)(k+2)}{2}$$

$$\Rightarrow \frac{k(k+1)}{2}+(k+1)=\frac{(k+1)(k+2)}{2}$$

$$Now,\ LHS=\frac{k(k+1)}{2}+(k+1)$$

$$=\frac{k(k+1)+2(k+1)}{2}$$

$$=\frac{(k+1)}{2}[k+2]=\frac{(k+1)(k+2)}{2}=R.H.S$$

Hence, $P(k+1)$ *is true.*

Hence, $P(n)$ *is true for all* $n\in N$.

2. $1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6},\ n\in N$

$P(n):1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$

Let $P(1):1^2=\frac{1(1+1)(2\times1+1)}{6}\Rightarrow 1=\frac{1(2)(3)}{6}\Rightarrow 1=1$

Hence, $P(1)$ *is true.*

Assume that $P(k)$ *be true.*

$P(k):1^2+2^2+3^2+...+k^2=\frac{k(k+1)(2k+1)}{6}$

To prove that $P(k+1)$ *is true.*

$P(k+1):1^2+2^2+3^2+...+k^2+(k+1)^2=\frac{(k+1)(k+2)[2(k+1)+1]}{6}$

$\Rightarrow P(k)+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$

$\Rightarrow \frac{k(k+1)(2k+1)}{6}+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$

Now, $LHS=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$

$=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$=\frac{(k+1)}{6}[k(2k+1)+6(k+1)]$

$=\frac{(k+1)}{6}[2k^2+k+6k+6]$

$=\frac{(k+1)}{6}[2k^2+7k+6]=\frac{(k+1)}{6}[2k^2+4k+3k+6]$

$=\frac{(k+1)}{6}[2k(k+2)+3(k+2)]$

$=\frac{(k+1)}{6}[(k+2)(2k+3)]$

$=\frac{(k+1)(k+2)(2k+3)}{6}=R.H.S$

Hence, $P(k+1)$ *is true.*

Hence, $P(n)$ *is true for all* $n\in N$.

3. $1.3+3.5+5.7+...+(2n-1)(2n+1)=\dfrac{n(4n^2+6n-1)}{3}$

$$P(n):1.3+3.5+5.7+...+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$$

$$P(n):1.3+3.5+5.7+...+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$$

$$Let\ \ P(1):1.3=\frac{1(4.1^2+6.1-1)}{3}\Rightarrow 3=\frac{1(9)}{3}\Rightarrow 3=3$$

Hence, $P(1)$ *is true.*

Assume that $P(k)$ *be true.*

$$P(k):1.3+3.5+5.7+...+(2k-1)(2k+1)=\frac{k(4k^2+6k-1)}{3}$$

To prove that $P(k+1)$ *is true.*

$$P(k+1):1.3+3.5+5.7+...+\left[2(k+1)-1\right]\left[2(k+1)+1\right]=\frac{(k+1)\left[4(k+1)^2+6(k+1)-1\right]}{3}$$

$$\Rightarrow P(k)+(2k+1)(2k+3)=\frac{(k+1)(4k^2+8k+4+6k+6-1)}{3}$$

$$\Rightarrow \frac{k(4k^2+6k-1)}{3}+4k^2+8k+3=\frac{(k+1)(4k^2+14k+9)}{3}$$

$$Now,\ LHS=\frac{k(4k^2+6k-1)}{3}+4k^2+8k+3$$

$$=\frac{4k^3+6k^2-k+12k^2+24k+9}{3}=\frac{4k^3+18k^2+23k+9}{3}$$

$$=\frac{4k^3+4k^2+14k^2+14k+9k+9}{3}$$

$$=\frac{4k^2(k+1)+14k(k+1)+9(k+1)}{3}$$

$$=\frac{(k+1)(4k^2+14k+9)}{3}=R.H.S$$

Hence, $P(k+1)$ *is true.*

Hence, $P(n)$ *is true for all* $n\in N$.

4. For every positive integer n, prove that $7^n - 3^n$ is divisible by 4.

$P(n): 7^n - 3^n$ is divisible by 4.

$$P(n): 7^n - 3^n \text{ is divisible by } 4$$
$$\text{Let } P(1): 7^1 - 3^1 = 4, \text{ is divisible by } 4$$
$$\text{Hence, } P(1) \text{ is true.}$$
$$\text{Assume that } P(k) \text{ be true.}$$
$$P(k): 7^k - 3^k \text{ is divisible by } 4$$
$$\Rightarrow 7^k - 3^k = 4d$$
$$\Rightarrow 7^k = 4d + 3^k \ldots\ldots\ldots\ldots..(1)$$
$$\text{To prove that } P(k+1) \text{ is true.}$$
$$P(k+1): 7^{k+1} - 3^{k+1} \text{ is divisible by } 4$$
$$\Rightarrow 7^k.7 - 3^k.3 = 7.7^k - 3.3^k = 7\left(4d + 3^k\right) - 3.3^k \quad \| \text{ From } (1)$$
$$= 4(7d) + 7.3^k - 3.3^k$$
$$= 4(7d) + 4.3^k = 4\left(7d + 3^k\right), \text{ is divisible by } 4$$
$$\text{Hence, } P(k+1) \text{ is true.}$$
$$\text{Hence, } P(n) \text{ is true for all } n \in N.$$

5. $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2n+1}{n^2}\right) = (n+1)^2$

$$P(n): \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2n+1}{n^2}\right) = (n+1)^2$$
$$\text{Let } P(1): 1+\frac{3}{1} = (1+1)^2 \Rightarrow 4 = 2^2 \Rightarrow 4 = 4$$
$$\therefore P(1) \text{ is true.}$$
$$\text{Assume that } P(k) \text{ be true.}$$
$$P(k): \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2k+1}{k^2}\right) = (k+1)^2$$
$$\text{To prove that } P(k+1) \text{ is true.}$$
$$P(k+1): P(k) \times \left(1+\frac{2(k+1)+1}{(k+1)^2}\right) = (k+1+1)^2$$

$$\Rightarrow (k+1)^2 \times \left(\frac{(k+1)^2 + 2(k+1) + 1}{(k+1)^2} \right) = (k+2)^2$$

$$LHS = (k+1)^2 \times \left(\frac{(k+1)^2 + 2(k+1) + 1}{(k+1)^2} \right)$$

$$= k^2 + 2k + 1 + 2k + 2 + 1 = k^2 + 4k + 4 = (k+2)^2 = RHS$$

Hence, $P(k+1)$ *is true.*

Hence, $P(n)$ *is true for all* $n \in N$.

6. $1 + 2 + 3 + ... + n < \frac{1}{8}(2n+1)^2$

Let $P(n) : 1 + 2 + 3 + ... + n < \frac{1}{8}(2n+1)^2$

$P(1) : 1 < \frac{1}{8}(2 \times 1 + 1)^2 \Rightarrow 1 < \frac{9}{8}$

$\therefore P(1)$ *is true.*

Assume that $P(k)$ *be true.*

$P(k) : 1 + 2 + 3 + ... + k < \frac{1}{8}(2k+1)^2$ ....(1)

To prove that $P(k+1)$ *is true.*

$P(k+1) : 1 + 2 + 3 + ... + k + (k+1) < \frac{1}{8}\left[2(k+1) + 1\right]^2$

$\Rightarrow 1 + 2 + 3 + ... + k + (k+1) < \frac{1}{8}(2k+3)^2$

In (1), *adding* $(k+1)$ *on both sides,we have*

$1 + 2 + 3 + ... + k + (k+1) < \frac{1}{8}(2k+1)^2 + (k+1)$

Now, $\frac{1}{8}(2k+1)^2 + (k+1) = \frac{1}{8}\left[(2k+1)^2 + 8(k+1)\right]$

$= \frac{1}{8}\left[4k^2 + 4k + 1 + 8k + 8\right]$

$= \frac{1}{8}\left[4k^2 + 12k + 9\right] = \frac{1}{8}(2k+3)^2$

$\therefore P(k+1)$ *is true.*

Hence, $P(n)$ *is true for all* $n \in N$.

7. $n(n+1)(n+2)$ *is a multiple of* 3

$Let\ P(n): n(n+1)(n+2)\ is\ a\ multiple\ of\ 6$

$P(1): 1(1+1)(1+2) = 6,\ is\ a\ multiple\ of\ 6$

$\therefore P(1)\ is\ true.$

$Assume\ that\ P(k)\ be\ true.$

$P(k): k(k+1)(k+2)\ is\ a\ multiple\ of\ 6$

$\Rightarrow k(k+1)(k+2) = 6M \quad .... (1)$

$To\ prove\ that\ P(k+1)\ is\ true.$

$P(k+1): (k+1)(k+2)(k+3)\ is\ a\ multiple\ of\ 6$

$$\Rightarrow (k+1)(k+2)(k+3) = (k+1)(k+2)k + (k+1)(k+2)3$$
$$= k(k+1)(k+2) + 3(k+1)(k+2)$$
$$= 6M + 3(2\lambda)$$

[$\because$ $(k+1)(k+2)$ are consecutive natural numbers, so either $(k+1)$ or $(k+2)$ is even.]

$$= 6(M+\lambda),\ divisible\ by\ 6.$$

$\therefore P(k+1)\ is\ true.$

$Hence,\ P(n)\ is\ true\ for\ all\ n \in N.$

EXTRA QUESTIONS.

Using principle of mathematical induction, prove the following:

1. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,\ n \in N$
2. $\sin\theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \dfrac{\sin\left(\frac{n+1}{2}\right)\theta . \sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)},\ n \in N$
3. $10^n + 3.4^{n+2} + 5$ is divisible by 9, $n \in N$.
4. ${}^nC_m \le n!,\ for\ all\ 1 \le m \le n,\ n \in N$
5. $3^n > 2^n,\ n \in N$
6. $(ab)^n = a^n b^n,\ n \in N$

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