CHAPTER 14 : MATHEMATICAL REASONING

38.1 INTRODUCTION

In this lesson, we shall learn about some basic ideas of mathematical reasoning and the process of reasoning especially in context of mathematics. In mathematical language, there are two kinds of reasoning. (i) Inductive reasoning and (ii) Deductive reasoning. We have already discussed the inductive reasoning in mathematical induction. Now, we shall discuss some fundamentals of deductive reasoning.

38.2 STATEMENT (OR PROPOSITION)

The basic unit involved in mathematical reasoning is a mathematical statement :

> A sentence is called a mathematically acceptable statement if it is either true or false but not both at the same time.

If a statement is true, we say that it is a valid statement. A false statement is known as an invalid statement.

- Consider the following two sentences :

  Three plus four is 6.

  Two plus three is 5.

  When we read these sentences, we immediately decide that the first sentence is wrong and second is correct. There is no confusion regarding these. In mathematics such sentences are called statements.

- Now consider the following sentence :

  Mathematics is fun.

  Mathematics is fun is true for those who like mathematics. But, for others, it may not be true. So, the given sentence is true or false both. Hence, it is not a statement.

- Consider the following sentences :

  (i) Moon revolves around the Earth.

  (ii) Every square is a rectangle.

  (iii) The Sun is a Star.

  (iv) Every rectangle is a square.

  (v) New Delhi is in Pakistan

  When we read these sentences, the first, second and third sentences are true but fourth and fifth are-false sentences. Hence, each of them is a statement.

- Consider the following sentences :

(i) Give me a glass of water

(ii) Switch on the light

(iii) Where are you going?

(iv) How are you?

(v) How beautiful!

(vi) May you live long!

(vii) Tomorrow is Wednesday

We can not decide the truth value of (i), (ii), (iii), (iv), (v), (vi) and (vii). Hence, they are not statements.

Example 38.1 Check whether the following sentences are statements. Give reasons for your answer.

(i) 12 is less than 16.

(ii) Every set is a finite set.

(iii) $x + 5 = 11$.

(iv) There is no rain without clouds.

(v) All integers are natural numbers.

(vi) How far is Agra form here?

(vii) Are you going to Kanpur?

(viii) All roses are white.

Solution : (i) This sentences is true, because $12 < 16$ (12 is less than 16). Hence, it is a statement.

(ii) This sentence is false, because there are sets which are not finite. Hence, it is a statement.

(iii) The sentence $x + 5 = 11$ is an open sentence. Its truth value cannot be confirmed unless we are given the value of x. Hence, it is not a statement.

(iv) It is scientifically established natural phenomenon that cloud is formed before it rains. Therefore, this sentence is always true. Hence, it is a statement.

(v) This sentence is false, because all integers are not natural numbers. So, it is a statement.

(vi) This sentence is a question (or interrogative sentence). Hence, it is not statement.

(vii) We can't have a truth value for it. So it is not a statement.

(viii) This sentence is false, because all roses are not white. Hence, it is a statement.

38.3 NEGATION OF A STATEMENT

"The denial of a statement is called the negation of the statement."

Let us consider the statement :

P : New Delhi is a city.

The negation of this statement is

It is not the case that New Delhi is a city.

or

It is false that New Delhi is a city

or

New Delhi is not a city.

> If p is statement, then the negation of p is also a statement and is denoted by $\sim p$, and read as 'not p'.

Example 38.2 Write the negation of the following statements :

(i) Sum of 2 and 3 is 6.

(ii) $\sqrt{7}$ is rational.

(iii) Australia is a continent.

(iv) The number is less than 5.

Solution : (i) P : Sum of 2 and 3 is 6.

~P : Sum of 2 and 3 is not 6.

(ii) q : $\sqrt{7}$ is rational

$\sim q$: $\sqrt{7}$ is not rational

or

It is false that $\sqrt{7}$ is rational

(iii) r : Australia is a continent

~r : Australia is not a continent

(iv) S : The number 8 is less than 5.

~S : The number 8 is not less than 5.

or

It is false that the number 8 is less than 5.

38.4 COMPOUND STATEMENTS

In mathematical reasoning, we generally come across two types of statements.

(1) **Simple Statements :** A statement which cannot be broken into two or more statements is called a simple statement. For example :

(i) Every set is a finite set

(ii) New Delhi is the capital of India

(iii) Roses are white

(iv) $\sqrt{2}$ is an irrational number

(v) The set of real numbers is an infinite set.

(2) **Compound Statement :** A statement that can be formed by combining two or more simple statements is called a compound statement.

For example :

(i) Mohan is very smart or he is very lucky. This statement is actually made up of two statements connected by "or".

p : Mohan is very smart.

q : Mohan is very Lucky.

(ii) Sun is bigger than earth and earth is bigger than moon.

This statement is made up of two simple statements connected by 'and'.

p : Sun is bigger than earth.

q : Earth is bigger than moon.

Example 38.3 Find the component statements of the following compound statements.

(i) The sky is blue and the grass is green.

(ii) All rational number are real and all real numbers are complex.

(iii) It is raining and it is cold.

(iv) $\sqrt{2}$ is a rational number or an irrational number.

Solution : (i) The component statements are

p : The sky is blue

q : The grass is green

The connecting word is 'and'.

(ii) The component statements are

p : All rational number are real

q : All real numbers are complex.

The connecting word is 'and'.

(iii) The component statements are

p : It is raining

q : It is cold.

The connecting word is 'and'

(iv) The component statements are

p : $\sqrt{2}$ is a rational number

q : $\sqrt{2}$ is an irrational number

The connecting word is 'or'

Example 38.4 Find the component statements of the following compound statements.

(i) 0 is positive number or negative number.

(ii) All prime numbers are either even or odd.

(iii) Chandigarh is the capital of Panjab and U.P.

(iv) 12 is multiple of 2, 3 and 4.

Solution : (i) The component statements are

P : 0 is a positive number

q : 0 is a negative number

The connecting word is 'or'.

(ii) The component statements are

p : All prime numbers are even numbers

q : All prime numbers are odd numbers

The connecting word is 'or'

(iii) The component statements are

p : Chandigarh is the capital of Panjab.

q : Chandigarh is the capital of U.P.

The connecting word is 'and'.

(iv) The component statements are

p : 12 is a multiple of 2

q : 12 is a multiple of 3

r : 12 is a multiple of 4

All the three statements are true. Here the connecting word is 'and'.

38.5 IMPLICATIONS

In this section, we shall discuss the implication "if then", "only if", and "if and only if".

The statements with "if then" are very common in mathematics. For example, consider the statement.

r : If you are born in some country, then you are a citizen of that country.

We observed that if corresponds to two statements p and q given by

p : you are born in some country

q : you are citizen of that country

p and q are two statements forming the implication "if p then q", then we denoted this implication by "$p \Rightarrow q$".

then, "if p then q" is the same as the following :

(i) If both p and q are true, then $p \Rightarrow q$ is also true.

(ii) If p is true and q is false, then $p \Rightarrow q$ is false.

(iii) If p is false and q is true, then $p \Rightarrow q$ is true

(iv) If p and q both are false, then $p \Rightarrow q$ is true.

Consider the following statements

If a number is a multiple if 9, then it is a multiple of 3.

It is an implication having antecedent(p) and consequent(q) as :

p : a number is multiple of 9

q : a number is multiple of 3.

the above statement says that

(i) p is sufficient condition for q.

this says that knowing that a number is a multiple of 9 is sufficient to conclude that it is a multiple of 3.

(ii) p only if q.

This states that a number is a multiple of 9 only if it is a multiple of 3.

(iii) q is necessary condition for p.

This says that when a number is a multiple of 9, it is necessarily a multiple of 3.

(iv) $\sim q$ implies $\sim p$.

This says that if a number is not a multiple of 3, then it is not a multiple of 9.

38.6 CONTRAPOSITIVE AND CONVERSE

Contrapositive : If p and q are two statements, then the contrapositive of the implication "if p then q" is "if $\sim q$, then $\sim p$".

Converse : If p and q are two statements, then the converse of the implication "if p-then q" is "if q-then p".

For example,

If a number is divisible by 9, then it is divisible by 3.

Its implication is as follows:

p : number is divisible by 9.

q : a number is divisible by 3.

The contrapositive of this statement is

If a number is not divisible by 3, it is not divisible by 9.

The converse of the statement is

If a number is divisible by 3, then it is divisible by 9.

38.7 IF AND ONLY IF IMPLICATION

If p and q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called if and only if implication and it is denoted by $p \Leftrightarrow q$.

For example,

A triangle is equilateral if and only if it is equiangular.

This is if and only if implication with the component statements

p : A triangle is equilateral

q : A triangle is equiangular

Example 38.5 Write the following statements in the form "if then".

(i) You get job implies that your credentials are good.

(ii) The banana trees will bloom if it stays warm for a month.

(iii) A quadrilateral is a parallelogram if its diagonals bisect each other.

Solution : (i) We know that "if p-then q" is equivalent to "$p \Rightarrow q$".

Then the given statement can be written as

"If you get a job, then your credentials are good".

(ii) We know that "if p-then q" is equivalent to "$p \Rightarrow q$"

The given statement can be written as

"If it stays warm for a month, then the banana trees will bloom".

(iii) The given statement can be written as

"If the diagonals of a quadrilateral bisect each other, then it is a parallelogram"

Example 38.6 Write the contrapositive of the following statements :

(i) If a triangle is equilateral, it is isosceles.

(ii) If you are born in India, then you are a citizen of India.

(iii) x is an even number implies that x is divisible by 4.

Solution : The contrapositive of these statements are

(i) If a triangle is not isosceles, then it is not equilateral.

(ii) If you are not a citizen of India, then you are not born in India.

(iii) If x is not divisible by 4, then x is not an even number.

Example 38.7 Write the converse of the following statements :

(i) If a number n is even, then n^2 is even.

(ii) If x is even number, then x is divisible by 4.

Solution : The converse of these statements are :

(i) If a number n^2 is even, then n is even.

(ii) If x is divisible by 4, then x is even.

Example 38.8 Given below are two pairs of statements. Combine these two statements using "if and only if".

(i) p : if a rectangle is a square, then all its four sides are equal.

q : if all the four sides of a rectangle are equal, then the rectangle is a square.

(ii) p : if the sum of digits of a number is divisible by 3, then the number is divisible by 3.

q : if a number is divisible by 3, then the sum of its digits is divisible by 3.

Solution : (i) A rectangle is a square if and only if all its four sides are equal.

(ii) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

38.8 VALIDATING STATEMENTS

In this section, we will discuss validity of statement. Checking the validity of statement means when it is true and when it is not true. The answer to these questions depend upon which of the special words and phrases "and", "or" and which of the implications "if and only if" "if-then", and which of the quantifiers "for every", "there exists", appear in the given statement.

Here, we shall discuss some techniques or rules to find when a statement is valid or true.

Rule 1 : Statements with "And"

If p and q are mathematical statements, then in order to show that the statement "p and q" is true, we follows the following steps :

Step-1 : Show that the statement p is true.

Step-2 : Show that the statement q is true.

Rule 2 : Statements with "or"

If p and q are mathematical statements, then in order to show that the statement "p or q" is true, one must consider the following.

Case 1 : Assuming that p is false, show that q must be true.

Case 2 : Assuming that q is false, show that p must be true.

Rule 3 : Validity of statements with "if-then".

If p and q are two mathematical statements, then to prove the statement "if p then q", we need to show that any one of the following case is true.

Case 1 : (Direct method)

By assuming that p is true, prove that q must true.

Case 2 : (Contrapositive method)

By assuming that q is false, prove that p must be false.

Rule 4 : Statements with "if and only if".

In order to prove the validity of the statement "p if and only if q" we need to show :

(i) If p is true then q is true.

(ii) If q is true the p is true.

Example 38.9 If p and q are two statements given by

p : 35 is multiple of 5

q : 35 is multiple of 6

Write the compound statement connecting these two statements with "and" and check the validity.

Solution : The compound statement "35 is multiple of 5 and 6. Since 35 is multiple of 5 but it is not multiple of 6. Therefore p is true but q is not true.

Example 38.10 Given below are two statements :

p : 35 is a multiple of 5

q : 35 is a multiple of 6

Write the compound statement connecting these two statements with "OR" and check its validity.

Solution : The compound statement is "35 is a multiple of 5 or 6."

By assuming that the statement q is false, then p is true.

Hence the compound statement is true i.e. valid.

Example 38.11 Check whether the following statement is true or not.

"If x and y are odd integers, then xy is an odd integer".

Solution : Let p and q be the statements given by

p : x and y are odd integers

q : xy is an odd integer

Then the given statement is

If p-then q.

Direct method : Let p be true, then,

p is true

$\Rightarrow$ x and y are odd integers

$\Rightarrow$ $x = 2m + 1$, $y = 2n + 1$ for some integers m, n

$\Rightarrow$ $xy = (2m + 1)(2n + 1)$

$\Rightarrow$ $xy = 2(2mn + m + n) + 1$

$\Rightarrow$ xy is an odd integer

$\Rightarrow$ q is true

Thus p is true $\Rightarrow$ q is true

Hence " if p-then q" is a true statement.

38.8.1 Contrapositive Method

Let q be not true. Then q is not true

$\Rightarrow$ xy is an even integer

$\Rightarrow$ either x is even or y is even or both x and y are even

$\Rightarrow$ p is not true

Thus q is false

$\Rightarrow$ p is false

Hence "If p-then q" is a true statement.

38.8.2 Validity of Statements by Contradiction

Here to check whether a statement p is true, we assume that p is not true i.e. $\sim p$ is true. Then we arrive at some result which contradicts our assumption. Therefore, we conclude that p is true.

Example 38.12 Verify by the method of contradiction p : $\sqrt{7}$ is irrational.

Solution : Let p be the statement given by p : $\sqrt{7}$ is irrational.

We assume that $\sqrt{7}$ is rational

$\Rightarrow$ $\sqrt{7} = \frac{a}{b}$, where a and b are integers having no common factor.

$\Rightarrow$ $7 = \frac{a^2}{b^2}$

$\Rightarrow$ $a^2 = 7b^2$

$\Rightarrow$ 7 divides a^2

$\Rightarrow$ 7 divides a

$\Rightarrow$ $a = 7c$ for some integer c

$\Rightarrow \quad a^2 = 49c^2$

$\Rightarrow \quad 7b^2 = 49c^2$

$\Rightarrow \quad b^2 = 7c^2$

$\Rightarrow$ 7 divides b^2

$\Rightarrow$ 7 divides b

Thus, 7 is common factor of both a and b. This contradicts that a and b have no common factor. So, our assumption $\sqrt{7}$ is rational is wrong. Hence the statement "$\sqrt{7}$ is irrational", is true.