Supporting Australian Mathematics Project

A guide for teachers – Years 11 and 12

Algebra and coordinate geometry: Module 11

Sequences and series

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Sequences and series – A guide for teachers (Years 11–12)

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Sequences and series

Assumed knowledge

The content of the modules:

- *Algebra review*
- *Functions I.*

Motivation

We encounter sequences at the very beginning of our mathematical experience. The list of even numbers

$$2,\ 4,\ 6,\ 8,\ 10,\ \dots$$

and the list of odd numbers

$$1,\ 3,\ 5,\ 7,\ 9,\ \dots$$

are examples. We can 'predict' what the 20th term of each sequence will be just by using common sense.

Another sequence of great historical interest is the **Fibonacci sequence**

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \dots$$

in which each term is the sum of the two preceding terms; for example, $55 = 21 + 34$. In this case it is somewhat more difficult to predict the 20th term, without listing all the previous ones.

Sequences arise in many areas of mathematics, including finance. For example, we can invest \$1000 at an interest rate of 5% per annum, compounded annually, and list the sequence consisting of the value of the investment each year:

$$\$1000,\ \$1050,\ \$1102.50,\ \$1157.63,\ \$1215.51,\ \dots$$

(rounded to the nearest cent).

Sequences can be either **finite** or **infinite**. For example,

$$2,\ 4,\ 6,\ 8,\ 10$$

is a finite sequence with five terms, whereas

$$2,\ 4,\ 6,\ 8,\ 10,\ \dots$$

continues without bound and is an infinite sequence. We usually use … to denote that the sequence continues without bound.

For a given infinite sequence, we can ask the questions:

- Can we find a formula for the general term of the sequence?
- Does the sequence have a limit, that is, do the numbers in the sequence get as close as we like to some number?

For example, we can see intuitively that the terms in the infinite sequence

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \dots$$

whose general term is $\frac{1}{n}$, are approaching 0 as n becomes very large.

A **finite series** arises when we add the terms of a finite sequence. For example,

$$2+4+6+8+\cdots+20$$

is the series formed from the sequence $2, 4, 6, 8, \dots, 20$.

An **infinite series** is the 'formal sum' of the terms of an infinite sequence. For example,

$$1+3+5+7+9+\cdots$$

is the series formed from the sequence of odd numbers. We can spot an interesting pattern in this series. The sum of the first two terms is 4, the sum of the first three terms is 9, and the sum of the first four terms is 16. So we guess that, in general, the sum of the first n terms is n^2.

For a given infinite series, we can ask the questions:

- Can we find a formula for the sum of the first n terms of the series?
- Does the series have a limit, that is, if we add the first n terms of the series, does this sum get as close as we like to some number as n becomes larger?

If it exists, this limit is often referred to as the **limiting sum** of the infinite series. In this module, we examine limiting sums for one special but commonly occurring type of series, known as a geometric series.

Sequences and series are very important in mathematics and also have many useful applications, in areas such as finance, physics and statistics.

Content

Sequences

The list of positive odd numbers

$$1,\ 3,\ 5,\ 7,\ 9,\ \dots$$

is an example of a typical **infinite sequence**. The dots indicate that the sequence continues forever, with no last term. We will use the symbol a_n to denote the n**th term** of a given sequence. Thus, in this example, $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and so on; the first term is $a_1 = 1$, but there is no last term.

The list of positive odd numbers less than 100

$$1,\ 3,\ 5,\ 7,\ \dots,\ 99$$

is an example of a typical **finite sequence**. The first term of this sequence is 1 and the last term is 99. This sequence contains 50 terms.

There are several ways to display a sequence:

- write out the first few terms
- give a formula for the general term
- give a recurrence relation.

Writing out the first few terms is not a good method, since you have to 'believe' there is some clearly defined pattern, and there may be many such patterns present. For example, if we simply write

$$1,\ 2,\ 4,\ \dots$$

then the next term might be 8 (powers of two), or possibly 7 (*Lazy Caterer's sequence*), or perhaps even 23 if there is some more complicated pattern going on. Hence, if the first few terms only are given, some rule should also be given as to how to uniquely determine the next term in the sequence.

A much better way to describe a sequence is to give a formula for the nth term a_n. This is also called a formula for the **general term**. For example,

$$a_n = 2n - 1$$

is a formula for the general term in the sequence of odd numbers $1, 3, 5, \dots$. From the formula, we can, for example, write down the 10th term, since $a_{10} = 2 \times 10 - 1 = 19$.

In some cases it is not easy, or even possible, to give an explicit formula for a_n. In such cases, it may be possible to determine a particular term in the sequence in terms of some of the preceding terms. This relationship is often referred to as a **recurrence**. For example, the sequence of positive odd numbers may be defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + 2, \ \text{for } n \geq 1.$$

The initial term is $a_1 = 1$, and the recurrence tells us that we need to add two to each term to obtain the next term.

The Fibonacci sequence comprises the numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots$$

where each term is the sum of the two preceding terms. This can be described by setting $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$, for $n \geq 1$.

Exercise 1

Consider the recurrence

$$a_1 = 2 \quad \text{and} \quad a_n = (a_{n-1})^2 + 1, \ \text{for } n \geq 2.$$

Write down the first five terms of this sequence.

The general term of a sequence can sometimes be found by 'pattern matching'.

Exercise 2

Give a formula for the general term of

a the sequence $2, 4, 6, 8, \dots$ of even numbers

b the sequence $1, 4, 9, 16, \dots$ of squares.

In general, however, finding a formula for the general term of a sequence can be difficult. Consider, for example, the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots.$$

We will discuss in the *History and applications* section how to show that the nth term of the Fibonacci sequence is given by

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

This is a very surprising result! (It is not even obvious that this formula will give an integer result for each n.) You might like to check that this formula works for $n = 1, 2, 3$.

Sequences can also be used to approximate real numbers. Thus, for example, the terms in the sequence

$$1,\ 1.4,\ 1.41,\ 1.414,\ 1.4142,\ \dots$$

give approximations to the real number $\sqrt{2}$.

Arithmetic sequences

We will limit our attention for the moment to one particular type of sequence, known as an **arithmetic sequence** (or **arithmetic progression**). This is a sequence of the form

$$a,\ a+d,\ a+2d,\ a+3d,\ \dots$$

where each term is obtained from the preceding one by adding a constant, called the **common difference** and often represented by the symbol d. Note that d can be positive, negative or zero.

Thus, the sequence of even numbers

$$2,\ 4,\ 6,\ 8,\ 10,\ \dots$$

is an arithmetic sequence in which the common difference is $d = 2$.

It is easy to see that the formula for the nth term of an arithmetic sequence is

$$a_n = a + (n-1)d.$$

Example

Find the formula for the nth term of the arithmetic sequence

1. $2,\ 5,\ 8,\ \dots$
2. $107,\ 98,\ 89,\ \dots.$

Solution

1. Here $a = 2$ and $d = 3$, so

$$a_n = 2 + (n-1) \times 3 = 3n - 1.$$

2. Here $a = 107$ and $d = -9$, so

$$a_n = 107 + (n-1) \times -9 = 116 - 9n.$$

Exercise 3

Find the nth term of the arithmetic sequence

$$\log_5 2,\ \log_5 4,\ \log_5 8,\ \dots.$$

We can also check whether a given number belongs to a given arithmetic sequence.

Example

Does the number 203 belong to the arithmetic sequence $3, 7, 11, \dots$?

Solution

Here $a = 3$ and $d = 4$, so $a_n = 3 + (n-1) \times 4 = 4n - 1$. We set $4n - 1 = 203$ and find that $n = 51$. Hence, 203 is the 51st term of the sequence.

Exercise 4

Show that 12 is not a term of the arithmetic sequence $210, 197, 184, \dots$.

Geometric sequences

A **geometric sequence** has the form

$$a,\ ar,\ ar^2,\ ar^3,\ \dots$$

in which each term is obtained from the preceding one by multiplying by a constant, called the **common ratio** and often represented by the symbol r. Note that r can be positive, negative or zero. The terms in a geometric sequence with negative r will oscillate between positive and negative.

The doubling sequence

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ \dots$$

is an example of a geometric sequence with first term 1 and common ratio $r = 2$, while

$$3,\ -6,\ 12,\ -24,\ 48,\ -96,\ \dots$$

is an example of a geometric sequence with first term 3 and common ratio $r = -2$.

It is easy to see that the formula for the nth term of a geometric sequence is

$$a_n = ar^{n-1}.$$

Example

Find the formula for the nth term of the geometric sequence

1. $2, 6, 18, \dots$
2. $486, 162, 54, \dots.$

Solution

1. Here $a = 2$ and $r = 3$, so $a_n = 2 \times 3^{n-1}$.
2. Here $a = 486$ and $r = \frac{1}{3}$, so $a_n = 486 \times \left(\frac{1}{3}\right)^{n-1}$.

Exercise 5

Find the nth term of the geometric sequence

$$\sqrt{6},\ 2\sqrt{3},\ 2\sqrt{6},\ \dots.$$

We can also check whether a given number belongs to a given geometric sequence.

Example

Does the number 48 belong to the geometric sequence

$$3072,\ 1536,\ 768,\ \dots?$$

Solution

Here $a = 3072$ and $r = \frac{1}{2}$, so $a_n = 3072 \times \left(\frac{1}{2}\right)^{n-1}$.

We set $3072 \times \left(\frac{1}{2}\right)^{n-1} = 48$. This gives $\left(\frac{1}{2}\right)^{n-1} = \frac{1}{64}$, that is, $2^{n-1} = 64 = 2^6$, and so $n = 7$.

Hence, 48 is the 7th term of the sequence.

Example

Does the number 6072 belong to the geometric sequence

$$3,\ -6,\ 12,\ -24,\ 48,\ \dots?$$

Solution

Here $a = 3$ and $r = -2$, so $a_n = 3 \times (-2)^{n-1}$.

We set $3 \times (-2)^{n-1} = 6072$. This gives $(-2)^{n-1} = 2024$.

But 2024 is not a power of 2, and so 6072 does not belong to the sequence.

Series

A **finite series** is the sum of the terms of a finite sequence. Thus, if

$$a_1,\ a_2,\ \dots,\ a_n$$

is a sequence of n terms, then the corresponding series is

$$a_1 + a_2 + \cdots + a_n.$$

The number a_k is referred to as the kth term of the series.

We often use the **sigma notation** for series. For example, if we have the series

$$2 + 4 + 6 + \cdots + 100$$

in which the kth term is given by $2k$, then we can write this series as

$$\sum_{k=1}^{50} 2k.$$

Note that the variable k here is a **dummy variable**. This means that we could also write the series as

$$\sum_{i=1}^{50} 2i \qquad \text{or} \qquad \sum_{j=1}^{50} 2j.$$

Exercise 6

By writing out the terms, find the sum

$$\sum_{k=1}^{n} \Big(\frac{1}{k} - \frac{1}{k+1} \Big).$$

An **infinite series** is the 'formal sum' of the terms of an infinite sequence:

$$a_1 + a_2 + a_3 + a_4 + \cdots .$$

For example, the sequence of odd numbers gives the infinite series $1+3+5+7+\cdots$.

We can sum an infinite series to a finite number of terms. The sum of the first n terms of an infinite series is often written as

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

This is sometimes called the ***n*th partial sum** of the infinite series.

Given a formula for the sum of the first n terms of a series, we can recover a formula for the nth term by a simple subtraction, as follows. Starting from

$$S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$$

$$S_{n-1} = a_1 + a_2 + \cdots + a_{n-1},$$

by subtracting we obtain

$$S_n - S_{n-1} = a_n.$$

For example, if the sum of the first n terms of a series is given by $S_n = n^2$, then the nth term is

$$a_n = S_n - S_{n-1} = n^2 - (n-1)^2 = 2n - 1.$$

So the terms form the sequence of odd numbers. Hence, we have found a formula for the sum of the first n odd numbers:

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

In general, it can be difficult to find a simple formula for the sum of a series to n terms. For the rest of this section, we restrict our attention to arithmetic and geometric series.

Arithmetic series

An **arithmetic series** is a series in which the terms form an arithmetic sequence. That is, each term is obtained from the preceding one by adding a constant.

The series

$$1+2+3+\cdots+n$$

is an arithmetic series with common difference 1. There is an easy way to find the sum of this series. We write the series forwards and then backwards:

$$\begin{array}{rccccccccccc} S_n = & 1 & + & 2 & + & 3 & + & \cdots & + & (n-1) & + & n \\ S_n = & n & + & (n-1) & + & (n-2) & + & \cdots & + & 2 & + & 1. \end{array}$$

Adding downwards in pairs, we obtain

$$2S_n = (1+n)+(2+n-1)+(3+n-2)+\cdots+(n-1+2)+(n+1).$$

Each of the n terms on the right-hand side simplifies to $n+1$. Thus $2S_n = n(n+1)$, and so we have shown that

$$1+2+3+\cdots+n = \frac{1}{2}n(n+1).$$

For example,

$$1+2+3+\cdots+100 = \frac{1}{2}\times 100\times 101 = 5050.$$

Legend has it that the famous mathematician Gauss discovered this at the age of nine!

This 'trick' works for any arithmetic series, and gives a formula for the sum S_n of the first n terms of an arithmetic series with first term $a_1 = a$ and last term $a_n = \ell$. The formula is

$$S_n = \frac{n}{2}(a+\ell).$$

Exercise 7

Use the method of writing the arithmetic series

$$a+(a+d)+(a+2d)+\cdots+(\ell-d)+\ell$$

forwards and backwards to derive the formula $S_n = \frac{n}{2}(a+\ell)$ given above.

Since the last term ℓ can be written as $a_n = a+(n-1)d$, where d is the common difference, we also have

$$\begin{aligned} S_n &= \frac{n}{2}(a+\ell) \\ &= \frac{n}{2}\big(a+a+(n-1)d\big) \\ &= \frac{n}{2}\big(2a+(n-1)d\big). \end{aligned}$$

Example

Find the formula for the sum of the first n terms of the arithmetic sequence

1. $2, 5, 8, \dots$
2. $107, 98, 89, \dots.$

Solution

1. Here $a = 2$ and $d = 3$, so

$$S_n = \frac{n}{2}\big(4 + (n-1)\times 3\big) = \frac{n}{2}\big(3n+1\big).$$

Alternatively, we can find the nth term of the sequence, which is $a_n = 3n-1$, and use the formula

$$S_n = \frac{n}{2}\big(a+\ell\big) = \frac{n}{2}\big(2+(3n-1)\big) = \frac{n}{2}\big(3n+1\big).$$

2. Here $a = 107$ and $d = -9$, so

$$S_n = \frac{n}{2}\big(2\times 107 + (n-1)\times -9\big) = \frac{n}{2}\big(223-9n\big).$$

For both parts of the previous example, we can substitute $n = 1$ and check this gives the first term of the series. Note that, since the formula for the sum is a quadratic, checking the three cases $n = 1$, $n = 2$, $n = 3$ is sufficient to prove that the answer is correct.

Exercise 8

Sum the arithmetic series

$$\log_2 3 + \log_2 9 + \log_2 27 + \cdots$$

to n terms.

Geometric series

A **geometric series** is a series in which the terms form a geometric sequence. That is, each term is obtained from the preceding one by multiplying by a constant.

For example,

$$2 + 8 + 32 + 128 + \cdots$$

is a geometric series with first term 2 and common ratio 4. The nth term is $a_n = 2\times 4^{n-1}$.

We can find a formula for the sum of the first n terms of this series, again using a little trick. We multiply the series by the common ratio 4 and subtract the original, as follows. Starting from

$$S_n = 2 + 8 + 32 + 128 + \cdots + 2 \times 4^{n-1}$$

$$4S_n = 8 + 32 + 128 + \cdots + 2 \times 4^{n-1} + 2 \times 4^n,$$

we subtract to obtain

$$4S_n - S_n = 2 \times 4^n - 2,$$

and so

$$S_n = \frac{1}{3}\big(2 \times 4^n - 2\big) = \frac{2(4^n - 1)}{3}.$$

This ‘trick’ works for any geometric series, and gives a formula for the sum S_n of the first n terms of a geometric series with first term a and common ratio r. The formula is

$$S_n = \frac{a(r^n - 1)}{r - 1}, \quad \text{for } r \neq 1.$$

Note that this can also be written as

$$S_n = \frac{a(1 - r^n)}{1 - r}, \quad \text{for } r \neq 1.$$

The second formula is often more convenient to use when r lies between -1 and 1.

In the case when $r = 1$, the sum of the series is clearly na, since all the terms are identical.

Exercise 9

Use the method of multiplying the geometric series

$$a + ar + ar^2 + \cdots + ar^{n-1}$$

by r and subtracting to derive the formula for S_n given above.

Example

Find the formula for the sum of the first n terms of the geometric sequence

1. $2, 6, 18, \dots$
2. $486, 162, 54, \dots.$

Solution

1. Here $a = 2$ and $r = 3$, so

$$S_n = \frac{2(3^n - 1)}{3 - 1} = 3^n - 1.$$

2. Here $a = 486$ and $r = \frac{1}{3}$, so

$$S_n = \frac{486\big(1 - (\frac{1}{3})^n\big)}{1 - \frac{1}{3}} = 729\big(1 - (\tfrac{1}{3})^n\big).$$

For both parts of the previous example, we can put $n = 1$ and check that we obtain the first term of the sequence.

Exercise 10

Find the sum to n terms of the geometric series

$$\sqrt{3} + 6 + 12\sqrt{3} + \cdots.$$

Summary

Arithmetic sequence	$a,\ a+d,\ a+2d,\ a+3d,\ \dots$ The nth term is $a_n = a + (n-1)d$, where a is the first term and d is the common difference.
Arithmetic series	$a + (a+d) + (a+2d) + (a+3d) + \cdots$ The sum of the first n terms is $$S_n = \frac{n}{2}\big(2a + (n-1)d\big),$$ where a is the first term and d is the common difference. This can also be written $S_n = \frac{n}{2}(a + \ell)$, where ℓ is the nth term a_n.
Geometric sequence	$a,\ ar,\ ar^2,\ ar^3,\ \dots$ The nth term is $a_n = ar^{n-1}$, where a is the first term and r is the common ratio.
Geometric series	$a + ar + ar^2 + ar^3 + \cdots$ The sum of the first n terms is $$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}, \quad \text{for } r \neq 1,$$ where a is the first term and r is the common ratio.

Means

It is a simple matter to find the average of two numbers. For example, the average of 6 and 10 is 8. When we do this, we are really finding a number x such that $6, x, 10$ forms an arithmetic sequence. In general, if the numbers a, x, b form an arithmetic sequence, then

$$x - a = b - x, \quad \text{giving} \quad x = \frac{a+b}{2}.$$

This is the average of a and b, also called the **arithmetic mean** (AM) of a and b.

Similarly, we can define the **geometric mean** (GM) of two positive numbers a and b to be the positive number x such that a, x, b forms a geometric sequence. In this case, we require

$$\frac{x}{a} = \frac{b}{x}, \quad \text{giving} \quad x = \sqrt{ab}.$$

Example

Find the arithmetic and geometric mean of

1 2, 18 **2** 3, 6.

Solution

1 $AM = \dfrac{2+18}{2} = 10, \quad GM = \sqrt{2 \times 18} = 6.$

2 $AM = \dfrac{3+6}{2} = \dfrac{9}{2}, \quad GM = \sqrt{3 \times 6} = \sqrt{18} = 3\sqrt{2}.$

Exercise 11

Suppose a and b are positive real numbers. First, take a line segment AB of length $a + b$ and mark a point X on it such that $AX = a$ (and so $XB = b$). Next, draw a semicircle, centred at the midpoint of AB, with AB as diameter. Finally, raise a perpendicular to AB at X to meet the semicircle at Y.

a Prove that the length XY is the geometric mean of a and b.

b By noting that the radius of the semicircle equals the arithmetic mean of a and b, deduce that

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

That is, the arithmetic mean of a and b is greater than or equal to their geometric mean.

Generalisations of these two means and examples of other types of means are discussed in the *Links forward* and *History and applications* sections.

Applications to finance

One of the many applications of sequences and series occurs in financial mathematics. Here we will briefly discuss compound interest and superannuation.

Compound interest

Compound interest was discussed in the TIMES module *Consumer arithmetic* (Year 9). We invest an amount $\$P$ at an interest rate r paid at the end of a prescribed interval for a period of n such intervals. Interest rates are usually quoted as percentages, and so an interest rate of 3% means that $r = 0.03$.

Let us here assume the time interval is one year. Hence, after one year, the investment has the value $P + rP = P(1+r)$. After the end of two years, the investment has the value $P(1+r) + rP(1+r) = P(1+r)^2$. We can see from this that, after n years, the investment will be worth $P(1+r)^n$.

These amounts form a geometric sequence with common ratio $1+r$, and the nth term is

$$P(1+r)^n.$$

Example

If \$1000 is invested at 4% p.a. compounded yearly, what is the value of the investment after ten years?

Solution

Here $P = 1000$, $r = 0.04$ and $n = 10$. Thus, after ten years, the investment is worth

$$1000(1+0.04)^{10} \approx \$1480.24.$$

Depreciation is closely related to compound interest. When a company, for example, buys a car for work-related purposes, it is able to claim the depreciation in the value of the car over time as a tax deduction.

If the car is initially worth $\$P$ and is depreciated at a rate of r per annum, then the value of the car after one year is $P - rP = P(1-r)$. After the end of two years, the car has value $P(1-r) - rP(1-r) = P(1-r)^2$. We can see from this that, after n years, the car will be worth $P(1-r)^n$.

Example

What is the value of a car after ten years, if it is initially worth \$30 000 and is depreciated at 6% p.a.?

Solution

Here $P = 30\,000$, $r = 0.06$ and $n = 10$. Thus, after ten years, the car is worth

$$30\,000\,(1-0.06)^{10} \approx \$16\,158.$$

Superannuation

Superannuation is a way of saving for retirement. Money is regularly invested over a long period of time and (compound) interest is paid. Suppose I invest in a superannuation scheme for 30 years which pays 6% per annum. I put \$3000 each year into the scheme, and (for the sake of simplicity) we will suppose that the interest is added yearly.

Thus, the first \$3000 will be invested for the full 30 years at 6% per annum. Hence, it will accrue to $\$3000 \times 1.06^{30}$. The next \$3000, deposited at the beginning of the second year, will be invested for 29 years and so will accrue to $\$3000 \times 1.06^{29}$, and so on. Writing the amounts from smallest to largest, the total value of my investment after 30 years is

$$A = 3000 \times 1.06 + 3000 \times 1.06^2 + \cdots + 3000 \times 1.06^{30}.$$

This is a geometric series with first term $a = 3000 \times 1.06$ and common ratio $r = 1.06$; the number of terms is $n = 30$. Using the formula for the sum of a finite geometric series, the final value of the investment is

$$A \approx \$251\,405.$$

This example illustrates both the value of regular saving and the power of compound interest. The \$90 000 invested becomes roughly \$251 400 over 30 years.

The limiting sum of a geometric series

We have seen that the sum of the first n terms of a geometric series with first term a and common ratio r is

$$S_n = \frac{a(1-r^n)}{1-r}, \quad \text{for } r \neq 1.$$

In the case when r has magnitude less than 1, the term r^n approaches 0 as n becomes very large. So, in this case, the sequence of partial sums $S_1, S_2, S_3, \dots$ has a limit:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}.$$

The value of this limit is called the **limiting sum** of the infinite geometric series. The values of the partial sums S_n of the series get as close as we like to the limiting sum, provided n is large enough.

The limiting sum is usually referred to as the **sum to infinity** of the series and denoted by S_∞. Thus, for a geometric series with common ratio r such that $|r| < 1$, we have

$$S_\infty = \lim_{n\to\infty} S_n = \frac{a}{1-r}.$$

Example

Find the limiting sum for the geometric series

1 $1 + \frac{1}{3} + \frac{1}{9} + \cdots$

2 $8 - 6 + \frac{9}{2} - \cdots$.

Solution

1 Here $a = 1$ and $r = \frac{1}{3}$, so the limiting sum exists and is equal to

$$S_\infty = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

2 Here $a = 8$ and $r = -\frac{3}{4}$, so the limiting sum exists and is equal to

$$S_\infty = \frac{8}{1+\frac{3}{4}} = \frac{32}{7}.$$

Exercise 12

Explain why the geometric series

$$1 + \frac{1}{1+\sqrt{2}} + \left(3 - 2\sqrt{2}\right) + \cdots$$

has a limiting sum, and find its value.

Exercise 13

By writing the recurring decimal $0.\overline{12} = 0.121212\ldots$ as

$$\frac{12}{10^2} + \frac{12}{10^4} + \cdots,$$

express $0.\overline{12}$ as a rational number in simplest form.

In general, the limiting sum of an infinite series $a_1 + a_2 + a_3 + \cdots$ is the limit, if it exists, of the sequence of partial sums $S_1, S_2, S_3, \ldots$, where

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

and so on. Infinite series are often written in the form

$$\sum_{n=1}^{\infty} a_n.$$

If the series has a limiting sum L, we say that the infinite series **converges** to L. This can be written as

$$\sum_{n=1}^{\infty} a_n = L \qquad \text{or} \qquad a_1 + a_2 + a_3 + \cdots = L.$$

These two expressions mean that the limit of the sequence of partial sums exists and is equal to the real number L; they can only be used if the infinite series converges.

Links forward

Use of induction

Some series are easy to sum by spotting patterns. For example, we have seen that the sum of the first n odd numbers is

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

On the other hand, the formula for the sum of the first n squares,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1),$$

is hardly obvious and requires a proof. While the Greeks had some very creative proofs of results such as this, the best approach is to give a proof using mathematical induction.

Telescoping series

Most series are neither arithmetic nor geometric. Some of these series can be summed by expressing the summand as a difference.

Example

1 Find the sum

$$\sum_{k=2}^{n} \frac{2}{k^2-1}.$$

2 Does the infinite series

$$\sum_{k=2}^{\infty} \frac{2}{k^2-1}$$

have a limiting sum? If so, what is its value?

Solution

1 We can factor k^2-1 and split the summand into

$$\frac{2}{k^2-1} = \frac{1}{k-1} - \frac{1}{k+1}.$$

Thus,

$$\begin{aligned} \sum_{k=2}^{n} \frac{2}{k^2-1} &= \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k+1} \right) \\ &= \sum_{k=2}^{n} \frac{1}{k-1} - \sum_{k=2}^{n} \frac{1}{k+1}. \end{aligned}$$

If we write out the terms of these two sums, we have

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-2} + \frac{1}{n-1}\right) - \left(\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1}\right).$$

Most of the terms cancel out (telescope), giving

$$\begin{aligned} \sum_{k=2}^{n} \frac{2}{k^2-1} &= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \\ &= \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}. \end{aligned}$$

2 Since the terms $\frac{1}{n}$ and $\frac{1}{n+1}$ go to zero as n goes to infinity, the series has a limiting sum of $\frac{3}{2}$.

The harmonic series

The **harmonic series** is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots.$$

There is no simple expression for the sum of the first n terms of this series. Does the series have a limiting sum? The following argument shows that the answer is no.

We can group the terms of the series as follows:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right) + \cdots.$$

Each term is greater than or equal to the last term in its bracket, and so we can write

$$\begin{aligned}
&1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\
&\qquad \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16}\right) + \cdots \\
&\qquad = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots,
\end{aligned}$$

which grows without bound. So the harmonic series does not have a limiting sum.

On the other hand, if we square each term and look at the series

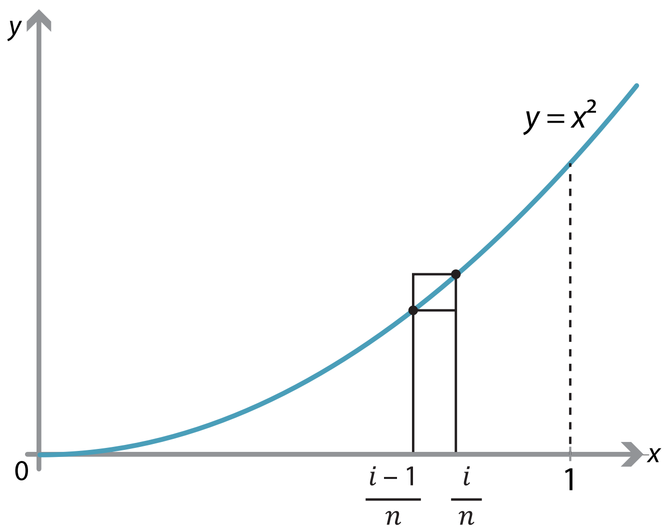
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots,$$

then it can be shown (although it is not all that easy) that this series has a limiting sum of $\frac{\pi^2}{6}$. This result was proven by Euler in the 18th century.

Connection with integration

Integration is used to find the area under a curve. We can approximate the area under the curve by rectangles, and add up the areas of the rectangles. This gives a finite series. By taking more rectangles of smaller width, we obtain a series that better approximates the area. We can define the area under the curve to be the limit of the sequence of sums, provided the limit exists.

Consider, for example, the function $f(x) = x^2$ on the interval $[0,1]$. We will look at the region R bounded by the curve, the x-axis and the line $x = 1$.



Approximating the area under $f(x) = x^2$ on the interval $[0,1]$.

Divide the interval $[0,1]$ into n equal subintervals of width $\frac{1}{n}$. For $i = 1,2,\dots,n$, we construct two rectangles U_i and L_i on the subinterval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, where

- the height of U_i is the *maximum* value of $f(x)$ on the subinterval, namely $\left(\frac{i}{n}\right)^2$
- the height of L_i is the *minimum* value of $f(x)$ on the subinterval, namely $\left(\frac{i-1}{n}\right)^2$.

We can obtain an *upper bound* for the area of R by finding the total area A_n of the n rectangles $U_1, U_2, \dots, U_n$:

$$\begin{aligned}
A_n &= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \dots + \left(\frac{n}{n}\right)^2\right] \\
&= \frac{1}{n^3}\left(1^2 + 2^2 + \dots + n^2\right) \\
&= \frac{1}{n^3} \times \frac{1}{6} n(n+1)(2n+1) \qquad \text{(sum of the first } n \text{ squares)} \\
&= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.
\end{aligned}$$

This is an **upper Riemann sum** for f on $[0,1]$.

Similarly, we can obtain a *lower bound* for the area of R by finding the total area B_n of the n rectangles $L_1, L_2, \dots, L_n$:

$$\begin{aligned} B_n &= \frac{1}{n}\left[\left(\frac{0}{n}\right)^2 + \left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right] \\ &= \frac{1}{n^3}\left(0^2 + 1^2 + \cdots + (n-1)^2\right) \\ &= \frac{1}{n^3} \times \frac{1}{6}(n-1)n(2n-1) \qquad \text{(sum of the first } n-1 \text{ squares)} \\ &= \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}. \end{aligned}$$

This is a **lower Riemann sum** for f on $[0,1]$.

Since $B_n \le \text{Area } R \le A_n$, we now have

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \le \text{Area } R \le \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

By taking the limit as $n \to \infty$, we see that the area of R is $\frac{1}{3}$. We can take this as the *definition* of the area of R.

More on means

The notions of arithmetic and geometric means can be extended to more than two numbers. For example, if a, b, c are positive numbers, their arithmetic and geometric means are defined to be

$$\frac{a+b+c}{3} \qquad \text{and} \qquad \sqrt[3]{abc},$$

respectively. In general, if $a_1, a_2, \dots, a_n$ are n positive real numbers, their arithmetic and geometric means are

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \qquad \text{and} \qquad \sqrt[n]{a_1 a_2 \dots a_n},$$

respectively.

Exercise 14

Suppose that $a_1, a_2, \dots, a_n$ is a sequence of positive real numbers and $b > 1$. Show that the logarithm (base b) of the geometric mean of these numbers is equal to the arithmetic mean of the numbers $\log_b a_1$, $\log_b a_2$, $\dots$, $\log_b a_n$.

The **harmonic mean** H of two positive numbers a and b is defined by

$$\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

It is the reciprocal of the average of the reciprocals of a and b. We can rearrange this equation as

$$H = \frac{2ab}{a+b}.$$

More generally, the harmonic mean H of positive numbers $a_1, a_2, \dots, a_n$ is defined by

$$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

The AM–GM inequality

Exercise 11 gave a geometric proof that the arithmetic mean of two positive numbers a and b is greater than or equal to their geometric mean. We can also prove this algebraically, as follows.

Since a and b are positive, we can define $x = \sqrt{a}$ and $y = \sqrt{b}$. Then

$$\begin{aligned} (x-y)^2 \geq 0 \quad &\Longrightarrow \quad x^2 + y^2 - 2xy \geq 0 \\ &\Longrightarrow \quad \frac{x^2 + y^2}{2} \geq xy \end{aligned}$$

and so

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

This is called the **AM–GM inequality**. Note that we have equality if and only if $a = b$.

Example

Find the range of the function $f(x) = x^2 + \dfrac{1}{x^2}$, for $x \neq 0$.

Solution

Using the AM–GM inequality,

$$f(x) = x^2 + \frac{1}{x^2} \geq 2\sqrt{x^2 \times \frac{1}{x^2}} = 2.$$

So the range of f is contained in the interval $[2, \infty)$. Note that $f(1) = 2$ and that $f(x) \to \infty$ as $x \to \infty$. Since f is continuous, it follows that, for each $y \geq 2$, there exists $x \geq 1$ with $f(x) = y$. Hence, the range of f is the interval $[2, \infty)$.

Exercise 15

a Find the arithmetic, geometric and harmonic means of $3,4,5$ and write them in ascending order.

b Prove that the harmonic mean of two positive real numbers a and b is less than or equal to their geometric mean.

The AM–GM inequality can be generalised as follows. If $a_1, a_2, \dots, a_n$ are n positive real numbers, then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \dots a_n}.$$

The next exercise provides a proof of this result.

Exercise 16

a Find the maximum value of the function $f(x) = \log_e x - x$, for $x > 0$. Hence, deduce that $\log_e x \leq x - 1$, for all $x > 0$.

b Let $a_1, a_2, \dots, a_n$ be positive real numbers and define $A = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$.

By successively substituting $x = \dfrac{a_i}{A}$, for $i = 1, 2, \dots, n$, into the inequality from part (a) and summing, show that

$$\log_e\Big(\frac{a_1 a_2 \dots a_n}{A^n}\Big) \leq 0.$$

c By exponentiating both sides of the inequality from part (b), derive the generalised AM–GM inequality.

History and applications

An application to film and video

In modern film and video, one has the power to vary the width and height of the images being filmed and replayed. The ratio between the width and height of an image is called its **aspect ratio**. This ratio is commonly expressed in the form $x : y$.

The most common aspect ratio used in movie theatres is $2.35 : 1$, while the aspect ratio traditionally used for television and video is $4 : 3 \approx 1.33 : 1$. It was found that the geometric mean of the numbers 2.35 and 1.33 provides a good compromise between the two different aspect ratios, distorting or cropping both in some sense equally. The geometric mean of 2.35 and 1.33 is approximately 1.77, and the ratio 1.77 : 1 corresponds approximately to 16 : 9. This is the aspect ratio adopted by the Society of Motion Picture and Television Engineers, and used for high-definition digital television.

Fibonacci numbers

The Fibonacci sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \dots$$

was first discussed in Europe by Leonardo of Pisa (whose nickname was Fibonacci) in the early 13th century, although the sequence can be traced back to about 200 BCE in Indian literature. This sequence has produced a large amount of literature and has connections to many branches of mathematics.

In the Fibonacci sequence, each term is the sum of the two preceding terms. So if a_n is the nth term, we can write

$$a_1 = a_2 = 1 \quad \text{and} \quad a_n = a_{n-1} + a_{n-2}, \ \text{ for } n \geq 3.$$

This is an example of a second-order linear recurrence relation.

A first-order linear recurrence such as $a_n = k a_{n-1}$, where k is a constant, is easily seen to have the solution $a_n = a_1 \times k^{n-1}$, which is an exponential. By taking $A = \frac{a_1}{k}$, we can write the solution as $a_n = A k^n$.

One approach to solving a second-order linear recurrence is to guess an exponential solution of the form $a_n = A k^n$, where A and k are non-zero constants. Substituting this into the recurrence for the Fibonacci sequence, we have

$$A k^n = A k^{n-1} + A k^{n-2}.$$

Dividing by $A k^{n-2}$, we see that k satisfies

$$k^2 = k + 1,$$

which has solutions $k = \frac{1}{2}(1 \pm \sqrt{5})$. Thus, we have found two exponential solutions

$$a_n = A_1 \times \left(\frac{1+\sqrt{5}}{2}\right)^n \qquad \text{and} \qquad a_n = A_2 \times \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The theory of recurrence relations tells us that the general solution of this recurrence is obtained by summing these two solutions:

$$a_n = A_1 \times \left(\frac{1+\sqrt{5}}{2}\right)^n + A_2 \times \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

We can now use the initial condition $a_1 = a_2 = 1$ to find that $A_1 = \frac{1}{\sqrt{5}}$ and $A_2 = -\frac{1}{\sqrt{5}}$. Finally, we have a formula for the nth Fibonacci number:

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Note that the number $\frac{1}{2}(1+\sqrt{5})$ that appears here is the **golden ratio**. One of many interesting facts about the Fibonacci sequence is that the only perfect squares in the sequence are 1 and 144.

The Greeks

The ancient Greek mathematicians were very interested in ratios. Indeed, much of their *arithmetic* was done geometrically using lengths and ratios. They found the link between ratios and music — hence the origin of the term *harmonic mean*.

The Greeks defined several different means. Less well known is the **Heronian mean** $N(a,b)$ of two positive real numbers a and b, which is the average of the three numbers a, b and their geometric mean. That is,

$$N(a,b) = \frac{a+b+\sqrt{ab}}{3}.$$

Notice that, for $a,b > 0$, we have

$$\frac{a+b+\sqrt{ab}}{3} \geq \frac{2\sqrt{ab}+\sqrt{ab}}{3} = \sqrt{ab}$$

and

$$\frac{a+b+\sqrt{ab}}{3} \leq \frac{a+b+\frac{a+b}{2}}{3} = \frac{a+b}{2},$$

giving

$$\sqrt{ab} \leq N(a,b) \leq \frac{a+b}{2}.$$

That is, the Heronian mean lies in between the geometric and arithmetic means.

Again, we can generalise and define the Heronian mean N of n positive real numbers $a_1, a_2, \dots, a_n$ to be

$$N = \frac{a_1 + a_2 + \cdots + a_n + \sqrt[n]{a_1 a_2 \dots a_n}}{n+1}.$$

The Greeks were also interested in various types of sequences and series. They of course knew the sequence of square numbers

$$1,\ 4,\ 9,\ 16,\ \dots$$

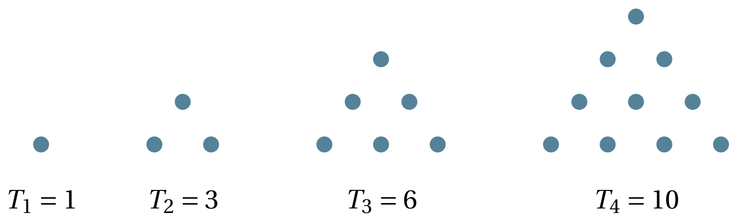
but they also introduced the sequence of **triangular numbers**

$$1,\ 3,\ 6,\ 10,\ \dots$$

given by $T_1 = 1$ and $T_n = T_{n-1} + n$, for $n \geq 2$. The general term of this sequence is

$$T_n = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

These numbers may be represented graphically, as follows.



The first four triangular numbers.

The Greeks used geometric techniques to show that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

and

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2 = (1+2+3+\cdots+n)^2.$$

Answers to exercises

Exercise 1

2, 5, 26, 677, 458 330.

Exercise 2

a $a_n = 2n$

b $a_n = n^2$.

Exercise 3

The sequence simplifies to

$$\log_5 2,\ 2\log_5 2,\ 3\log_5 2,\ \dots$$

and so the general term is $a_n = n\log_5 2$.

Exercise 4

Here $a = 210$ and $d = -13$, so the general term is given by $a_n = 210 - 13(n-1) = 223 - 13n$. The equation $223 - 13n = 12$ has solution $n = \frac{211}{13}$, which is not positive integer. Hence 12 is not a term in the sequence.

Exercise 5

We have $a = \sqrt{6}$ and $r = \dfrac{2\sqrt{3}}{\sqrt{6}} = \sqrt{2}$. Thus $a_n = \sqrt{6}(\sqrt{2})^{n-1} = \sqrt{3}(\sqrt{2})^n$.

Exercise 6

$$\begin{aligned}\sum_{k=1}^{n}\Big(\frac{1}{k} - \frac{1}{k+1}\Big) &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{4}\Big) + \cdots + \Big(\frac{1}{n} - \frac{1}{n+1}\Big)\\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1}.\end{aligned}$$

Exercise 7

Writing the series forwards and backwards, we have

$$\begin{array}{rccccccccccc} S_n = & a & + & (a+d) & + & (a+2d) & + & \cdots & + & (\ell - d) & + & \ell \\ S_n = & \ell & + & (\ell - d) & + & (\ell - 2d) & + & \cdots & + & (a+d) & + & a. \end{array}$$

Adding in pairs gives

$$2S_n = (a+\ell) + (a+\ell) + \cdots + (a+\ell) = n(a+\ell).$$

Hence, $S_n = \frac{n}{2}(a+\ell)$.

Exercise 8

Here $a = \log_2 3$ and $d = \log_2 3$, so

$$S_n = \frac{n}{2}\big(2\log_2 3 + (n-1)\log_2 3\big) = \frac{1}{2}n(n+1)\log_2 3.$$

Exercise 9

We have

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^{n-1}\\ rS_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n. \end{aligned}$$

Subtracting gives

$$\begin{aligned} rS_n - S_n = ar^n - a \quad &\Longrightarrow \quad S_n(r-1) = a(r^n - 1)\\ &\Longrightarrow \quad S_n = \frac{a(r^n-1)}{r-1}, \quad \text{provided } r \neq 1. \end{aligned}$$

Exercise 10

We have $a = \sqrt{3}$ and $r = \dfrac{6}{\sqrt{3}} = 2\sqrt{3}$. Hence, $S_n = \dfrac{\sqrt{3}\big((2\sqrt{3})^n - 1\big)}{2\sqrt{3}-1}$.

Exercise 11

a The triangles AXY and YXB are similar. (They have equal angles, as $\angle AYB = 90°$.) Thus

$$\frac{a}{XY} = \frac{XY}{b} \quad \Longrightarrow \quad XY = \sqrt{ab}.$$

Hence, the length XY is the geometric mean of a and b.

b Let C be the midpoint of AB. Let D be the point where the perpendicular to AB at C cuts the semicircle. Then CD is the radius of the semicircle, and so $CD = \frac{1}{2}(a+b)$. Clearly, $CD \geq XY$, and therefore $\frac{1}{2}(a+b) \geq \sqrt{ab}$.

Exercise 12

The common ratio is

$$r = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1,$$

and so $-1 < r < 1$. Hence, the geometric series has a limiting sum, given by

$$S_\infty = \frac{1}{1-(\sqrt{2}-1)} = \frac{1}{2-\sqrt{2}} = 1 + \frac{\sqrt{2}}{2}.$$

Exercise 13

We can write the decimal $0.\overline{12}$ as

$$0.\overline{12} = 0.12121212\ldots = \frac{12}{10^2} + \frac{12}{10^4} + \frac{12}{10^6} + \cdots,$$

which is a geometric series with $a = \frac{12}{10^2}$ and $r = \frac{1}{10^2}$. The limiting sum is

$$0.\overline{12} = \left(\frac{12}{10^2}\right) \times \left(\frac{1}{1-\frac{1}{10^2}}\right) = \frac{4}{33}.$$

Exercise 14

The geometric mean of $a_1, a_2, \ldots, a_n$ is $G = \sqrt[n]{a_1 a_2 \ldots a_n}$. So

$$\begin{aligned}\log_b G &= \log_b\big((a_1 a_2 \ldots a_n)^{\frac{1}{n}}\big)\\ &= \frac{1}{n}\log_b(a_1 a_2 \ldots a_n)\\ &= \frac{1}{n}\big(\log_b a_1 + \log_b a_2 + \cdots + \log_b a_n\big),\end{aligned}$$

which is the arithmetic mean of $\log_b a_1$, $\log_b a_2$, $\ldots$, $\log_b a_n$.

Exercise 15

a The arithmetic mean is $AM = 4$ and the geometric mean is $GM = \sqrt[3]{60} \approx 3.91$. The harmonic mean HM satisfies

$$\frac{1}{HM} = \frac{1}{3}\Big(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\Big) = \frac{47}{180} \qquad \Longrightarrow \qquad HM = \frac{180}{47} \approx 3.83.$$

So in this case $HM \le GM \le AM$.

b Using the AM–GM inequality, we have $a + b \ge 2\sqrt{ab}$, and so

$$HM = \frac{2ab}{a+b} \le \frac{2ab}{2\sqrt{ab}} = \sqrt{ab} = GM.$$

Exercise 16

a Let $f(x) = \log_e x - x$, for $x > 0$. Then $f'(x) = \frac{1}{x} - 1$, so the only stationary point is at $x = 1$. We could use a sign diagram to check that $f(x)$ has a maximum at $x = 1$. The maximum is $f(1) = -1$. So, for all $x > 0$, we have $\log_e x - x \le -1$, giving $\log_e x \le x - 1$.

b Define $A = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$. Substituting $x = \frac{a_1}{A}$, $x = \frac{a_2}{A}$, …, $x = \frac{a_n}{A}$ into the inequality $\log_e x \le x - 1$ from part (a) gives

$$\begin{aligned}
\log_e\Big(\frac{a_1}{A}\Big) &\le \frac{a_1}{A} - 1 \\
\log_e\Big(\frac{a_2}{A}\Big) &\le \frac{a_2}{A} - 1 \\
&\vdots \\
\log_e\Big(\frac{a_n}{A}\Big) &\le \frac{a_n}{A} - 1.
\end{aligned}$$

Adding, we have

$$\log_e\Big(\frac{a_1}{A}\Big) + \log_e\Big(\frac{a_2}{A}\Big) + \cdots + \log_e\Big(\frac{a_n}{A}\Big) \le \frac{1}{A}(a_1 + a_2 + \cdots + a_n) - n.$$

Hence,

$$\log_e\Big(\frac{a_1 a_2 \dots a_n}{A^n}\Big) \le n - n = 0.$$

c Exponentiating both sides of the inequality above gives

$$\frac{a_1 a_2 \dots a_n}{A^n} \le 1 \qquad \Longrightarrow \qquad a_1 a_2 \dots a_n \le A^n,$$

from which we have

$$(a_1 a_2 \dots a_n)^{\frac{1}{n}} \le A = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

which is the desired result.

0 1 2 3 4 5 6 7 8 9 10 11 12