R.K.MALIK'S
NEWTON CLASSES
JEE (MAIN & ADV.), MEDICAL + BOARD, NDA, IX & X

CHAPTER 11 : CONIC SECTIONS

While cutting a carrot you might have noticed different shapes shown by the edges of the cut. Analytically you may cut it in three different ways, namely

(*i*) Cut is parallel to the base (see Fig.16.1)

(*ii*) Cut is slanting but does not pass through the base (see Fig.16.2)

(*iii*) Cut is slanting and passes through the base (see Fig.16.3)

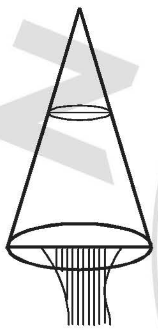


Fig.16.1

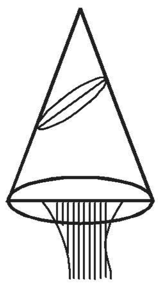


Fig.16.2

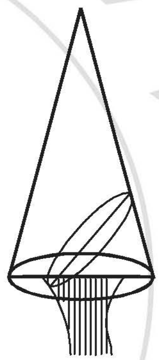


Fig.16.3

The different ways of cutting, give us slices of different shapes.

In the first case, the slice cut represent a circle which we have studied in previous lesson.

In the second and third cases the slices cut represent different geometrical curves, which we shall study in this lesson.

OBJECTIVES

After studying this leson, you will be able to :

- recognise a circle, parabola, ellipse and hyperbola as sections of a cone;
- recognise the parabola, ellipse and hyperbola as certain loci;
- identify the concept of eccentricity, directrix, focus and vertex of a conic section;
- identify the standard equations of parabola, ellipse and hyperbola;
- find the equation of a parabola, ellipse and hyperbola given its directrix and focus.

EXPECTED BACKGROUND KNOWLEDGE

- Basic knowledge of coordinate Geometry
- Various forms of equation of a straight line
- Equation of a circle in various forms

16.1 CONIC SECTION

In the introduction we have noticed the various shapes of the slice of the carrot. Since the carrot is conical in shape so the section formed are sections of a cone. They are therefore called conic sections.

Mathematically, a conic section is the locus of a point P which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed line.

The fixed point is called the ***focus*** and is usually denoted by S.

The fixed straight line is called the ***Directrix.***

The straight line passing through the focus and perpendicular to the directrix is called the ***axis.***

The constant ratio is called the ***eccentricity*** and is denoted by e.

What happens when

(i) $e<1$ (ii) $e=1$ (iii) $e>1$

In these cases the conic section obtained are known as ellipse, parabola and hyperbola respectively.

In this lesson we shall study about ellipse, parabola, and hyperbola.

16.2 ELLIPSE

Recall the cutting of slices of a carrot. When we cut it obliquely, slanting without letting the knife pass through the base, what do we observe?

You might have come across such shapes when you cut a boiled egg vertically.

The slice thus obtained represents an ellipse. Let us define the ellipse mathematically as follows:

> ***"An ellipse is the locus of a point which moves in a plane such that its distance from a fixed point bears a constant ratio to its distance from a fixed line and this ratio is less than unity".***

16.2.1 STANDARD EQUATION OF AN ELLIPSE

Let S be the focus, ZK be the directrix and P be a moving point. Draw SK perpendicular from S on the directrix. Let e be the eccentricity.

Divide SK internally and externally at A and A' (on KS produced) repectively in the ratio e : 1, as e<1.

$SA = e.AK \qquad \cdots (1)$

and $SA' = e.A'K \qquad \cdots (2)$

Since A and A' are points such that their distances from the focus bears a constant ratio e ($e < 1$) to their respective distances from the directrix and so they lie on the ellipse. These points are called vertices of the ellipse.

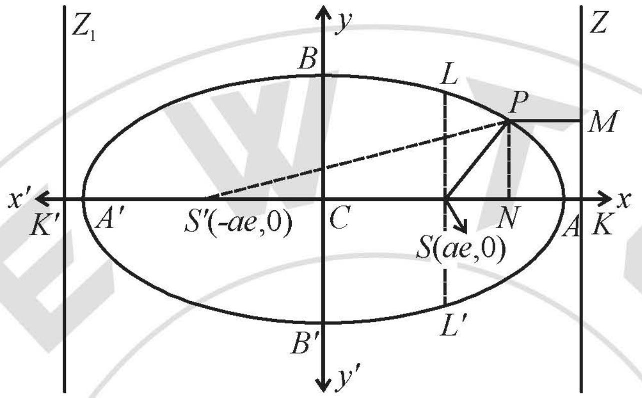


Fig. 16.4

Let AA' be equal to $2a$ and C be its mid point, i.e., $CA = CA' = a$

The point C is called the centre of the ellipse.

Adding (1) and (2), we have

$$SA + SA' = e.AK + e.A'K$$

A

or $\quad AA' = e(CK - CA + A'C + CK)$ or $\quad 2a = e.2CK$ or $\quad CK = \dfrac{a}{e} \qquad \cdots (3)$

Subtracting (1) from (2), we have

$$SA' - SA = e(A'K - AK)$$

or $\quad (SC + CA') - (CA - CS) = e.A'A$

or $\quad 2\,CS = e.2a$ or $\quad CS = ae \qquad \cdots (4)$

Let us choose C as origin, CAX as x-axis and CY, a line perpendicular to CX as y-axis.

$\therefore$ Coordinates of S are then $(ae, 0)$ and equation of the directrix is $x = \dfrac{a}{e}$

Let the coordinates of the moving point P be (x, y). Join SP, draw $PM \perp ZK$.

By definition $SP = e.PM$ or $\quad SP^2 = e^2 .\ PM^2$

or $\quad SN^2 + NP^2 = e^2.(NK)^2$ or $\quad (CN - CS)^2 + NP^2 = e^2.(CK - CN)^2$

or $\quad (x-ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$ or $x^2(1-e^2) + y^2 = a^2(1-e^2)$

or $\quad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$ [On dividing by $a^2(1-e^2)$]

Putting $a^2(1-e^2) = b^2$, we have the standard form of the ellipse as, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Major axis : The line joining the two vertices A' and A, i.e., $A'A$ is called the major axis and its length is $2a$.

Minor axis : The line passing through the centre perpendicular to the major axis, i.e., BB' is called the minor axis and its length is $2b$.

Principal axis : The two axes together (major and minor) are called the principal axes of the ellipse.

Latus rectum : The length of the line segment LL' is called the latus rectum and it is given by

$$\frac{2b^2}{a}$$

Equation of the directrix : $x = \pm\frac{a}{e}$

Eccentricity : e is given by $e^2 = 1 - \frac{b^2}{a^2}$

Example 16.1 Find the equation of the ellipse whose focus is (1,–1), eccentricity $e = \frac{1}{2}$ and the directrix is $x - y = 3$.

Solution : Let $P(h,k)$ be any point on the ellipse then by the definition, its distance from the focus = e. Its distance from directrix or $SP^2 = e^2.PM^2$

(M is the foot of the perpendicular drawn from P to the directrix).

or $\quad (h-1)^2 + (k+1)^2 = \frac{1}{4}\left(\frac{h-k-3}{\sqrt{1+1}}\right)^2$

or $\quad 7(h^2+k^2) + 2hk - 10h + 10k + 7 = 0$

$\therefore$ The locus of P is, $7(x^2+y^2) + 2xy - 10x + 10y + 7 = 0$

which is the required equation of the ellipse.

Example 16.2 Find the eccentricity, coordinates of the foci and the length of the axes of the ellipse $3x^2 + 4y^2 = 12$

Solution : The equation of the ellipse can be written in the following form, $\frac{x^2}{4} + \frac{y^2}{3} = 1$

On comparing this equation with that of the standard equation of the ellipse, we have $a^2 = 4$ and $b^2 = 3$, then

(i) $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow e = \frac{1}{2}$

(ii) coordinates of the foci are $(1,0)$ and $(-1,0)$

[$\because$ The coordinate are ($\pm$ ae, 0)]

(*iii*) Length of the major axes $2a = 2 \times 2 = 4$ and

length of the minor axis = $2b = 2 \times \sqrt{3} = 2\sqrt{3}.$

16.3 PARABOLA

Recall the cutting of slice of a carrot. When we cut obliquely and letting the knife pass through the base, what do we observe?

Also when a batsman hits the ball in air, have you ever noticed the path of the ball?

Is there any property common to the edge of the slice of the carrot and the path traced out by the ball in the example cited above?

Yes, the edge of such a slice and path of the ball have the same shape which is known as a parabola. Let us define parabola mathematically.

"A parabola is the locus of a point which moves in a plane so that its distance from a fixed point in the plane is equal to its distance from a fixed line in the plane."

16.3.1 STANDARD EQUATION OF A PARABOLA

Let S be the fixed point and ZZ' be the directrix of the parabola. Draw SK perpendicular to ZZ'. Bisect SK at A.

Since $SA = AK$, by the definition of the parabola A lies on the parabola. A is called the vertex of the parabola.

Take A as origin, AX as the x-axis and AY perpendicular to AX through A as the y-axis.

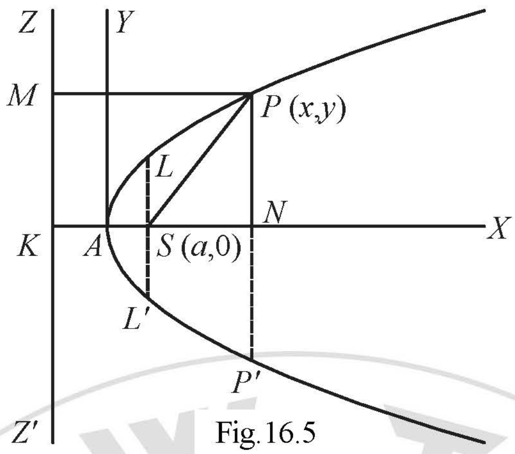


Fig. 16.5

Let $KS = 2a$ $\quad \therefore AS = AK = a$

$\therefore$ The coordinates of A and S are (0,0) and (a,0) respectively.

Let $P(x,y)$ be any point on the parabola. Draw $PN \perp AS$ produced

$\therefore$ $AN = x$ and $NP = y$

Join SP and draw $PM \perp ZZ'$

$\therefore$ By definition of the parabola

$SP = PM$ or $\quad SP^2 = PM^2$

or $\quad (x-a)^2 + (y-0)^2 = (x+a)^2 \quad [\because PM = NK = NA + AK = x + a]$

or $\quad (x-a)^2 - (x+a)^2 = -y^2$ or $\quad y^2 = 4ax$

which is the ***standard equation*** of the parabola.

Note : *In this equation of the parabola*

(i) Vertex is (0,0)

(ii) Focus is (a,0)

(iii) Equation of the axis is $y = 0$

(iv) Equation of the directrix is $x + a = 0$

(v) Latus rectum = $4a$

16.3.2 OTHER FORMS OF THE PARABOLA

What will be the equation of the parabola when

(i) focus is ($-a$,0) and directrix is $x - a = 0$

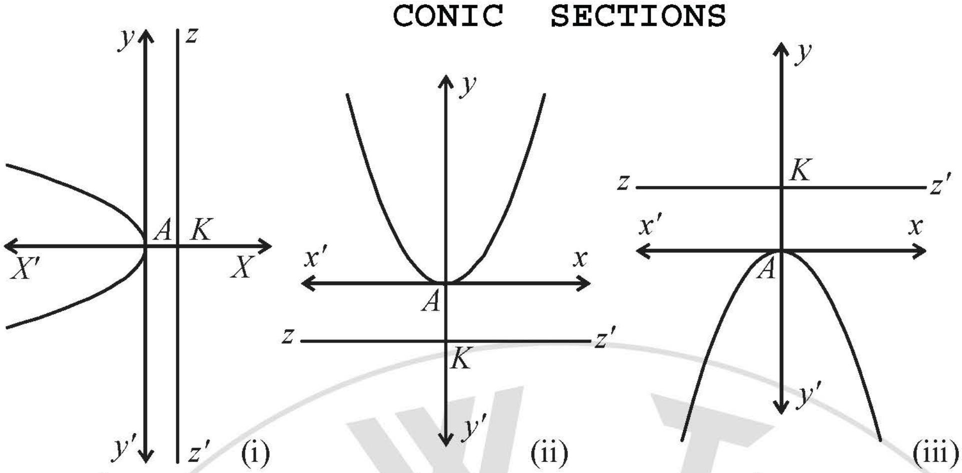
(ii) focus is (0,a) and directrix is $y + a = 0$,

(iii) focus is (0, $-a$) and directrix is $y - a = 0$?

It can easily be shown that the equation of the parabola with above conditions takes the following forms:

(i) $y^2 = -4ax$ $\quad$ (ii) $x^2 = 4ay$ $\quad$ (iii) $x^2 = -4ay$

The figures are given below for the above equations of the parabolas.



$y^2 = -4ax$ $\quad$ $x^2 = 4ay$ $\quad$ $x^2 = -4ay$

Fig. 16.6

Corresponding results of above forms of parabolas are as follows:

Forms	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Coordinates of vertex	(0,0)	(0,0)	(0,0)	(0,0)
Coordinates of focus	$(a,0)$	$(-a,0)$	$(0,a)$	$(0,-a)$
Coordinates of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Coordinates of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
length of Latus rectum	$4a$	$4a$	$4a$	$4a$

Example 16.3 Find the equation of the parabola whose focus is the origin and whose directrix is the line $2x + y - 1 = 0$.

Solution : Let $S(0,0)$ be the focus and ZZ' be the directrix whose equation is $2x + y - 1 = 0$

Let $P(x, y)$ be any point on the parabola.

Let PM be perpendicular to the directrix (See Fig. 16.5)

$\therefore$ By definition $SP = PM$ or $SP^2 = PM^2$

or
$$x^2 + y^2 = \frac{(2x + y - 1)^2}{\left(\sqrt{2^2 + 1}\right)^2}$$

or $\quad 5x^2 + 5y^2 = 4x^2 + y^2 + 1 + 4xy - 2y - 4x$ or $\quad x^2 + 4y^2 - 4xy + 2y + 4x - 1 = 0.$

NEWTON CLASSES

Example 16.4 Find the equation of the parabola, whose focus is the point (2, 3) and whose directrix is the line $x - 4y + 3 = 0$.

Solution : Given focus is $S(2,3)$; and the equation of the directrix is $x - 4y + 3 = 0$.

$\therefore$ As in the above example, $(x-2)^2 + (y-3)^2 = \left\{\dfrac{x-4y+3}{\sqrt{1^2+4^2}}\right\}^2$

$\Rightarrow$ $16x^2 + y^2 + 8xy - 74x - 78y + 212 = 0$

16.4 HYPERBOLA

Hyperbola is the locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its distance from a fixed straight line in the same plane is greater than one. In other words hyperbola is the conic in which eccentricity is greater than unity. The fixed point is called focus and the fixed straight line is called directrix.

Equation of Hyperbola in Standard from :

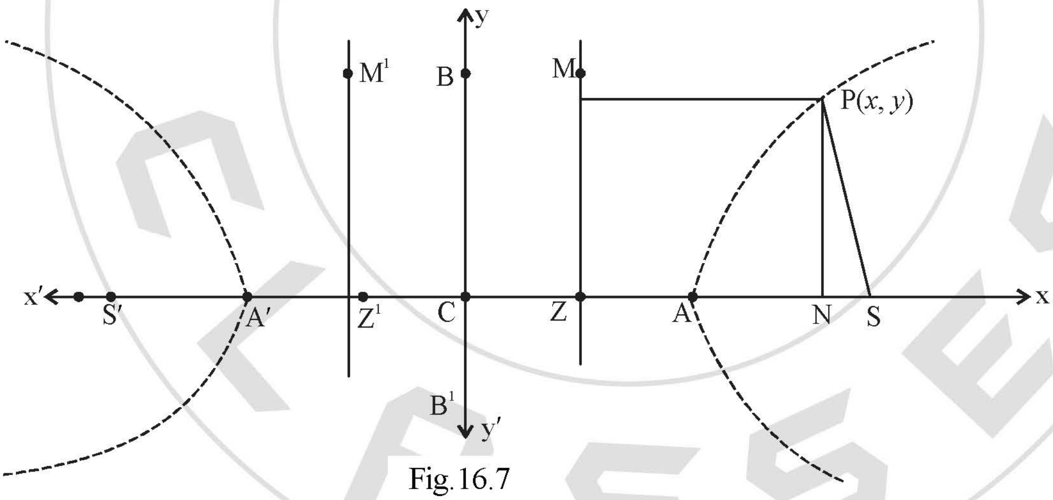


Fig.16.7

Let S be the focus and ZM be the directrix. Draw SZ perpendicular from S on directix we can divide SZ both internally and externally in the ratio $e : 1$ $(e > 1)$. Let the points of division be A and A′ as shown in the above figure. Let C be the mid point of AA′. Now take CZ as the x-axis and the perpendicular at C as y-axis.

Let $\quad AA' = 2a$

Now $\quad \dfrac{SA}{AZ} = e\ (e > 1)$ and $\dfrac{SA'}{A'Z} = e\ (e > 1)$.

i.e. $SA = eAZ$ ...(i)

i.e. $SA' = eA'Z$ ...(ii)

Adding (i) and (ii) we get

$$SA + SA' = e(AZ + A'Z)$$

$$(CS - CA) + (CS + CA') = eAA'$$

$$\Rightarrow \quad 2CS = e.2a \; (\because CA = CA')$$

$$\Rightarrow \quad CS = ae$$

Hence focus point is $(ae, 0)$.

Subtracting (i) from (ii) we get

$$SA' - SA = e(A'Z - AZ)$$

i.e. $AA' = e\left[(CZ + CA') - (CA - CZ)\right]$

i.e. $AA' = e[2CZ] \quad (\because CA' = CA)$

i.e. $2a = e(2CZ)$

$$\Rightarrow \quad CZ = \frac{a}{e}$$

$\therefore$ Equation of directrix is $x = \frac{a}{e}$.

Let $P(x, y)$ be any point on the hyperbola, PM and PN be the perpendiculars from P on the directrix and x-axis respectively.

Thus, $\frac{SP}{PM} = e \quad \Rightarrow SP = ePM$

$\Rightarrow \quad (SP)^2 = e^2(PM)^2$

i.e. $(x - ae)^2 + (y - 0)^2 = e^2\left(x - \frac{a}{e}\right)^2$

i.e. $x^2 + a^2e^2 - 2aex + y^2 = e^2\left(\frac{e^2x^2 + a^2 - 2aex}{e^2}\right)$

i.e. $x^2 + a^2e^2 + y^2 = e^2x^2 + a^2$

i.e. $(e^2 - 1)x^2 - y^2 = a^2(e^2 - 1)$

i.e. $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$

Let $$a^2(e^2-1) = b^2$$

$$\therefore \quad \boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1}$$

Which is the equation of hyperbola in standard from.

- Now let S′ be the image of S and Z′M′ be the image of ZM w.r.t y-axis. Taking S′ as focus and Z′M′ as directrix, it can be seen that the corresponding equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Hence for every hyperbola, there are two foci and two directrices.
- We have $b^2 = a^2(e^2-1)$ and $e > 1$

$$\Rightarrow \qquad e = \sqrt{\frac{a^2+b^2}{a^2}}$$

- If we put $y = 0$ in the equation of hyperbola we get
  $x^2 = a^2 \Rightarrow x = \pm a$
  $\therefore$ Hyperbola cuts x-axis at A(a, 0) and A′($-a$, 0).
- If we put $x = 0$ in the equation of hyperbola we get

  $y^2 = -b^2 \Rightarrow y = \pm\sqrt{-1}.b = \pm ib$

  Which does not exist in the cartesian plane.
  $\therefore$ Hyperbola does not interesct y-axis.
- AA′ = $2a$, along the x-axis is called **transverse axis** of the hyperbola and BB′ = $2b$, along y-axis is called **conjugate axis** of the hyperbola. Notice that hyperbola does not meet its conjugate axis.
- As in case of ellipse, hyperbola has two foci

  S(ae, 0), S′($-ae$, 0) and two directrices $x = \pm\frac{a}{e}$.
- C is called the centre of hyperbola.
- Latus rectum of hyperbola is a line segment perpendicular to the transverse axis through any of the foci and whose end points lie on the hyperbola. As in ellipse, it can be proved that the length of the latus rectum of hyperbola is $\frac{2b^2}{a}$.
- Hyperbola is symmetric about both the axes.
- Foci of hyperbola are always on transverse axis. It is the positive term whose denominator gives the transverse axis. For example $\frac{x^2}{9}-\frac{y^2}{16}=1$ has transverse axis along x-

axis and length of transverse axis is 6 units. While $\frac{y^2}{25}-\frac{x^2}{16}=1$ has transverse axis along y-axis of length 10 unit.

- The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axis of given hyperbola, is called the conjugate hyperbola of the given hyperbola. This equation is of the form $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$.

In this case : Transverse axis is along y-axis and conjugate axis is along x-axis.

- Length of transverse axis = $2b$.
- Length of conjugate axis = $2a$
- Length of latus rectum = $\frac{2a^2}{b}$.
- Equations of directrices $y=\pm\frac{b}{e}$.
- Vertices $(0, \pm b)$
- Foci $(0, \pm be)$
- Centre $(0, 0)$
- Eccentricity $(e)=\sqrt{\frac{b^2+a^2}{b^2}}$.

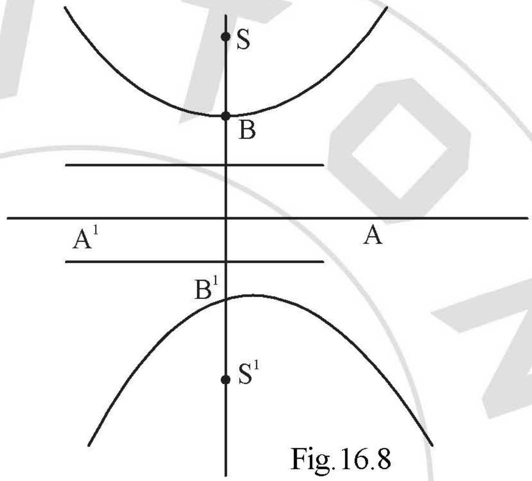


Fig.16.8

16.4.1 RECTANGULAR HYPERBOLA

If in a hyperbola the length of the transverse axis is equal to the length of the conjugate axis, then the hyperbola is called a rectangular hyperbola.

Its equation is $x^2-y^2=a^2$ or $y^2-x^2=b^2$ ($\because a=b$)

In this case $e=\sqrt{\frac{a^2+a^2}{a^2}}$ or $\sqrt{\frac{b^2+b^2}{b^2}}=\sqrt{2}$

i.e. the eccentricity of rectangualr hyperbola is $\sqrt{2}$.

Example 16.5 For the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$, find the following (i) Eccentricity (ii) Foci (iii) Vertices (iv) Directrices (v) Length of transverse axis (vi) Length of conjugate axis (vii) Length of latus rectum (viii) Centre.

Solution : Here $a^2=16$ and $b^2=9$, $\Rightarrow a=4$ and $b=3$.

(i) Eccentricity (e) = $\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{16+9}{16}}=\frac{5}{4}$

(ii) Foci = $(\pm\ ae, 0)=\left(\pm\frac{4.5}{4},0\right)=(\pm 5,0)$

(iii) Vertices = $(\pm a, 0)=(\pm 4, 0)$

(iv) Directrices $x = \pm \frac{a}{e} \Rightarrow x = \pm \frac{4}{5/4} \Rightarrow x = \pm \frac{16}{5}$.

(v) Length of transverse axis = $2a = 2 \times 4 = 8$.

(vi) Length of conjugate axis = $2a = 2 \times 3 = 6$

(vii) Length of latus rectum = $\frac{2b^2}{a} = \frac{2 \times 9}{4} = \frac{9}{2}$.

(viii) Centre = (0, 0)

Example 16.6 Find the equation of hyperbola with vertices (± 2, 0) and foci (± 3, 0)

Solution : Here $a = 2$ and $ae = 3$.

$\therefore \quad e = 3/2.$

We know that $\quad b^2 = a^2(e^2 - 1)$

$\Rightarrow \quad b^2 = 4\left(\frac{9}{4} - 1\right) = 5$

$\therefore$ Equation of hyperbola is $\frac{x^2}{4} - \frac{y^2}{5} = 1$.

Example 16.7 For hyperbola $\frac{y^2}{9} - \frac{x^2}{27} = 1$, find the following :

(i) Eccentricity (ii) Centre (iii) Foci (iv) Vertices (v) Directrices (vi) Length of transverse axis (vii) Length of conjugate axis (viii) Latus rectum.

Solution : Here $b^2 = 9$ and $a^2 = 27 \Rightarrow b = 3$ and $a = 3\sqrt{3}$.

(i) $e = \sqrt{\frac{27+9}{9}} = \sqrt{4} = 2$. (ii) Centre = (0, 0)

(iii) Foci = $(0, \pm be) = (0, \pm 3.2) = (0, \pm 6)$.

(iv) Vertices = $(0, \pm b) = (0, \pm 3)$.

(v) Directrices, $y = \pm \frac{b}{e} \Rightarrow y = \pm \frac{3}{2}$.

(vi) Length of transverse axis = $2b = 2 \times 3 = 6$

(vii) Length of conjugate axis = $2a = 2 \times 3\sqrt{3} = 6\sqrt{3}$

(viii) Length of latus rectum = $\frac{2a^2}{b} = \frac{2 \times 27}{3} = 18$.

NEWTON CLASSES

LET US SUM UP

- **Conic Section**

"A conic section is the locus of a point P which moves so that its distance from a fixed point is always in a constant ratio to its perpendicular distance from a fixed straight line".

(*i*) **Focus :** The fixed point is called the focus.

(*ii*) **Directrix :** The fixed straight line is called the directrix.

(*iii*) **Axis :** The straight line passing through the focus and pependicular to the directrix is called the axis.

(*iv*) **Eccentricity :** The constant ratio is called the eccentricity.

- (*v*) **Latus Rectum :** The double ordinate passing through the focus and parallel to the directrix is known as latus rectum. (In Fig.16.5 LSL' is the latus rectum).

- **Standard Equation of the Ellipse is :** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(*i*) Major axis = $2a$ (*ii*) Minor axis = $2b$

(*iii*) Equation of directrix is $x=\pm\frac{a}{e}$ (*iv*) Foci : $(\pm ae, 0)$

(*v*) Eccentricity, i.e., e is given by $e^2=1-\frac{b^2}{a^2}$ vi Latus Reotam $=\frac{2b^2}{a}$

- **Standard Equation of the Parabola is :** $y^2=4ax$

(*i*) Vertex is (0,0) (*ii*) Focus is $(a,0)$

(*iii*) Axis of the parabola is $y=0$ (*iv*) Directrix of the parabola is $x+a=0$

(*v*) Latus rectum = $4a$.

- **OTHER FORMS OF THE PARABOLA ARE**

  (*i*) $y^2 = -4ax$ (concave to the left). (*ii*) $x^2 = 4ay$ (concave upwards). (*iii*) $x^2 = -4ay$ (concave downwards).

- Equation of hyperbola having transverse axis along x-axis and conjugate axis along y-axis is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

  For this hyperbola (i) $e = \sqrt{\frac{a^2 + b^2}{a^2}}.$

  (ii) Centre = (0, 0) (iii) Foci = ($\pm ae$, 0)

  (iv) Vetrices = ($\pm a$, 0)(v) Length of latus rectum = $\frac{2b^2}{a}$

  (vi) Length of transverse axis = $2a$

  (vii) Length of conjugate axis = $2b$

  (viii) Equations of directrixes are given by $x = \pm \frac{a}{e}.$

- Equations of hyperbola having transverse axis along y-axis and conjugate axis along x-axis is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$

  For this hyperbola :

  (i) Vertices = (0, $\pm b$)(ii) Centre = (0, 0)

  (iii) Foci = (0, $\pm be$) (iv) $e = \sqrt{\frac{a^2 + b^2}{b^2}}$

  (v) Length of latus rectum = $\frac{2a^2}{b}.$

  (vi) Length of transverse axis = $2b$.

  (vii) Length of conjugate axis = $2a$.

  (viii) Equations of directrixes are given by $y = \pm \frac{b}{e}.$

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