Exercise 8.1

Question 1:

Expand the expression $(1-2x)^5$

Answer

By using Binomial Theorem, the expression $(1-2x)^5$ can be expanded as

$$(1-2x)^5$$
$$= {}^5C_0(1)^5 - {}^5C_1(1)^4(2x) + {}^5C_2(1)^3(2x)^2 - {}^5C_3(1)^2(2x)^3 + {}^5C_4(1)^1(2x)^4 - {}^5C_5(2x)^5$$
$$= 1 - 5(2x) + 10(4x^2) - 10(8x^3) + 5(16x^4) - (32x^5)$$
$$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$$

Question 2:

Expand the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$

Answer

By using Binomial Theorem, the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$ can be expanded as

$$\left(\frac{2}{x}-\frac{x}{2}\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2$$
$$- {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$$
$$= \frac{32}{x^5} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) - 10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32}$$
$$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}$$

Question 3:

Expand the expression $(2x-3)^6$

Answer

By using Binomial Theorem, the expression $(2x-3)^6$ can be expanded as

$$\begin{aligned}(2x-3)^6 &= {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2 - {}^6C_3(2x)^3(3)^3\\ &\quad + {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5 + {}^6C_6(3)^6\\ &= 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27)\\ &\quad + 15(4x^2)(81) - 6(2x)(243) + 729\\ &= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729\end{aligned}$$

Question 4:

Expand the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

Answer

By using Binomial Theorem, the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ can be expanded as

$$\begin{aligned}\left(\frac{x}{3}+\frac{1}{x}\right)^5 &= {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2\\ &\quad + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5\\ &= \frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5}\\ &= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}\end{aligned}$$

Question 5:

Expand $\left(x+\frac{1}{x}\right)^6$

Answer

By using Binomial Theorem, the expression $\left(x+\frac{1}{x}\right)^6$ can be expanded as

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0(x)^6 + {}^6C_1(x)^5\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2$$
$$+{}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$
$$= x^6 + 6(x)^5\left(\frac{1}{x}\right) + 15(x)^4\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right) + 15(x)^2\left(\frac{1}{x^4}\right) + 6(x)\left(\frac{1}{x^5}\right) + \frac{1}{x^6}$$
$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Question 6:

Using Binomial Theorem, evaluate $(96)^3$

Answer

96 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, 96 = 100 – 4

$$\therefore (96)^3 = (100-4)^3$$
$$= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)(4)^2 - {}^3C_3(4)^3$$
$$= (100)^3 - 3(100)^2(4) + 3(100)(4)^2 - (4)^3$$
$$= 1000000 - 120000 + 4800 - 64$$
$$= 884736$$

Question 7:

Using Binomial Theorem, evaluate $(102)^5$

Answer

102 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 102 = 100 + 2

$$\therefore (102)^5 = (100+2)^5$$
$$= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3$$
$$+ {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$$
$$= (100)^5 + 5(100)^4(2) + 10(100)^3(2)^2 + 10(100)^2(2)^3 + 5(100)(2)^4 + (2)^5$$
$$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$$
$$= 11040808032$$

Question 8:

Using Binomial Theorem, evaluate $(101)^4$

Answer

101 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 101 = 100 + 1

$$\therefore (101)^4 = (100+1)^4$$
$$= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4$$
$$= (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4$$
$$= 100000000 + 4000000 + 60000 + 400 + 1$$
$$= 104060401$$

Question 9:

Using Binomial Theorem, evaluate $(99)^5$

Answer

99 can be written as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 99 = 100 – 1

$$\begin{aligned}
\therefore (99)^5 &= (100-1)^5 \\
&= {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 \\
&\quad + {}^5C_4(100)(1)^4 - {}^5C_5(1)^5 \\
&= (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 5(100) - 1 \\
&= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1 \\
&= 10010000500 - 500100001 \\
&= 9509900499
\end{aligned}$$

Question 10:

Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Answer

By splitting 1.1 and then applying Binomial Theorem, the first few terms of $(1.1)^{10000}$ can be obtained as

$$\begin{aligned}
(1.1)^{10000} &= (1+0.1)^{10000} \\
&= {}^{10000}C_0 + {}^{10000}C_1(1.1) + \text{Other positive terms} \\
&= 1 + 10000 \times 1.1 + \text{Other positive terms} \\
&= 1 + 11000 + \text{Other positive terms} \\
&> 1000
\end{aligned}$$

Hence, $(1.1)^{10000} > 1000$

Question 11:

Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$.

Answer

Using Binomial Theorem, the expressions, $(a + b)^4$ and $(a - b)^4$, can be expanded as

$$(a+b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$$

$$(a-b)^4 = {}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4$$

$$\therefore (a+b)^4 - (a-b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$$
$$-\left[{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4\right]$$
$$= 2\left({}^4C_1a^3b + {}^4C_3ab^3\right) = 2\left(4a^3b + 4ab^3\right)$$
$$= 8ab\left(a^2 + b^2\right)$$

By putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4 = 8\left(\sqrt{3}\right)\left(\sqrt{2}\right)\left\{\left(\sqrt{3}\right)^2 + \left(\sqrt{2}\right)^2\right\}$$
$$= 8\left(\sqrt{6}\right)\{3+2\} = 40\sqrt{6}$$

Question 12:

Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$.

Answer

Using Binomial Theorem, the expressions, $(x + 1)^6$ and $(x - 1)^6$, can be expanded as

$$(x+1)^6 = {}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_4x^2 + {}^6C_5x + {}^6C_6$$

$$(x-1)^6 = {}^6C_0x^6 - {}^6C_1x^5 + {}^6C_2x^4 - {}^6C_3x^3 + {}^6C_4x^2 - {}^6C_5x + {}^6C_6$$

$$\therefore (x+1)^6 + (x-1)^6 = 2\left[{}^6C_0x^6 + {}^6C_2x^4 + {}^6C_4x^2 + {}^6C_6\right]$$
$$= 2\left[x^6 + 15x^4 + 15x^2 + 1\right]$$

By putting $x = \sqrt{2}$, we obtain

$$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6 = 2\left[\left(\sqrt{2}\right)^6 + 15\left(\sqrt{2}\right)^4 + 15\left(\sqrt{2}\right)^2 + 1\right]$$
$$= 2\left(8 + 15\times4 + 15\times2 + 1\right)$$
$$= 2\left(8 + 60 + 30 + 1\right)$$
$$= 2(99) = 198$$

Question 13:

Show that $9^{n+1}-8n-9$ is divisible by 64, whenever *n* is a positive integer.

Answer

In order to show that $9^{n+1}-8n-9$ is divisible by 64, it has to be proved that,

$9^{n+1}-8n-9=64k$, where *k* is some natural number

By Binomial Theorem,

$$(1+a)^m = {}^mC_0 + {}^mC_1a + {}^mC_2a^2 + \dots + {}^mC_ma^m$$

For *a* = 8 and *m* = *n* + 1, we obtain

$$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + \dots + {}^{n+1}C_{n+1}(8)^{n+1}$$

$$\Rightarrow 9^{n+1} = 1 + (n+1)(8) + 8^2\left[{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \dots + {}^{n+1}C_{n+1}(8)^{n-1}\right]$$

$$\Rightarrow 9^{n+1} = 9 + 8n + 64\left[{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \dots + {}^{n+1}C_{n+1}(8)^{n-1}\right]$$

$$\Rightarrow 9^{n+1} - 8n - 9 = 64k, \text{ where } k = {}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \dots + {}^{n+1}C_{n+1}(8)^{n-1} \text{ is a natural number}$$

Thus, $9^{n+1}-8n-9$ is divisible by 64, whenever *n* is a positive integer.

Question 14:

Prove that $\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$.

Answer

By Binomial Theorem,

$$\sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r = (a+b)^n$$

By putting *b* = 3 and *a* = 1 in the above equation, we obtain

$$\sum_{r=0}^{n} {}^nC_r\, (1)^{n-r} (3)^r = (1+3)^n$$

$$\Rightarrow \sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$$

Hence, proved.

Exercise 8.2

Question 1:

Find the coefficient of x^5 in $(x + 3)^8$

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^5 occurs in the $(r + 1)^{th}$ term of the expansion $(x + 3)^8$, we obtain

$$T_{r+1} = {}^8C_r (x)^{8-r} (3)^r$$

Comparing the indices of x in x^5 and in T_{r+1}, we obtain

$r = 3$

Thus, the coefficient of x^5 is ${}^8C_3 (3)^3 = \frac{8!}{3!5!} \times 3^3 = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2.5!} \cdot 3^3 = 1512$

Question 2:

Find the coefficient of a^5b^7 in $(a - 2b)^{12}$

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that a^5b^7 occurs in the $(r + 1)^{th}$ term of the expansion $(a - 2b)^{12}$, we obtain

$$T_{r+1} = {}^{12}C_r (a)^{12-r} (-2b)^r = {}^{12}C_r (-2)^r (a)^{12-r} (b)^r$$

Comparing the indices of a and b in a^5 b^7 and in T_{r+1}, we obtain

$r = 7$

Thus, the coefficient of a^5b^7 is

$${}^{12}C_7 (-2)^7 = -\frac{12!}{7!5!} \cdot 2^7 = -\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8.7!}{5 \cdot 4 \cdot 3 \cdot 2.7!} \cdot 2^7 = -(792)(128) = -101376$$

Question 3:

Write the general term in the expansion of $(x^2 - y)^6$

Answer

It is known that the general term T_{r+1} {which is the $(r + 1)^{th}$ term} in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 – y^6)$ is

$$T_{r+1} = {}^6C_r \left(x^2\right)^{6-r} \left(-y\right)^r = \left(-1\right)^r\, {}^6C_r . x^{12-2r} . y^r$$

Question 4:

Write the general term in the expansion of $(x^2 – yx)^{12}$, $x \neq 0$

Answer

It is known that the general term T_{r+1} {which is the $(r + 1)^{th}$ term} in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the general term in the expansion of$(x^2 – yx)^{12}$ is

$$T_{r+1} = {}^{12}C_r \left(x^2\right)^{12-r} \left(-yx\right)^r = \left(-1\right)^r\, {}^{12}C_r . x^{24-2r} . y^r . x^r = \left(-1\right)^r\, {}^{12}C_r . x^{24-r} . y^r$$

Question 5:

Find the 4^{th} term in the expansion of $(x – 2y)^{12}$.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, the 4^{th} term in the expansion of $(x – 2y)^{12}$ is

$$T_4 = T_{3+1} = {}^{12}C_3 \left(x\right)^{12-3} \left(-2y\right)^3 = \left(-1\right)^3 \cdot \frac{12!}{3!9!} \cdot x^9 \cdot \left(2\right)^3 \cdot y^3 = -\frac{12 \cdot 11 \cdot 10}{3 \cdot 2} \cdot \left(2\right)^3 x^9 y^3 = -1760 x^9 y^3$$

Question 6:

Find the 13^{th} term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18},\ x \neq 0$.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$.

Thus, 13th term in the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}$ is

$$T_{13}=T_{12+1}={}^{18}C_{12}(9x)^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$=(-1)^{12}\frac{18!}{12!6!}(9)^6(x)^6\left(\frac{1}{3}\right)^{12}\left(\frac{1}{\sqrt{x}}\right)^{12}$$

$$=\frac{18\cdot17\cdot16\cdot15\cdot14\cdot13.12!}{12!.6\cdot5\cdot4\cdot3\cdot2}\cdot x^6.\left(\frac{1}{x^6}\right).3^{12}\left(\frac{1}{3^{12}}\right) \qquad \left[9^6=\left(3^2\right)^6=3^{12}\right]$$

$$=18564$$

Question 7:

Find the middle terms in the expansions of $\left(3-\frac{x^3}{6}\right)^7$

Answer

It is known that in the expansion of $(a + b)^n$, if n is odd, then there are two middle terms, namely, $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term.

Therefore, the middle terms in the expansion of $\left(3-\frac{x^3}{6}\right)^7$ are $\left(\frac{7+1}{2}\right)^{th}=4^{th}$ term and $\left(\frac{7+1}{2}+1\right)^{th}=5^{th}$ term

$$T_4=T_{3+1}={}^7C_3(3)^{7-3}\left(-\frac{x^3}{6}\right)^3=(-1)^3\frac{7!}{3!4!}.3^4\cdot\frac{x^9}{6^3}$$

$$=-\frac{7\cdot6\cdot5.4!}{3\cdot2.4!}\cdot3^4\cdot\frac{1}{2^3\cdot3^3}\cdot x^9=-\frac{105}{8}x^9$$

$$T_5=T_{4+1}={}^7C_4(3)^{7-4}\left(-\frac{x^3}{6}\right)^4=(-1)^4\frac{7!}{4!3!}(3)^3\cdot\frac{x^{12}}{6^4}$$

$$=\frac{7\cdot6\cdot5.4!}{4!.3\cdot2}\cdot\frac{3^3}{2^4\cdot3^4}\cdot x^{12}=\frac{35}{48}x^{12}$$

Thus, the middle terms in the expansion of $\left(3-\frac{x^3}{6}\right)^7$ are $-\frac{105}{8}x^9$ and $\frac{35}{48}x^{12}$.

Question 8:

Find the middle terms in the expansions of $\left(\frac{x}{3}+9y\right)^{10}$

Answer

It is known that in the expansion $(a + b)^n$, if n is even, then the middle term is $\left(\frac{n}{2}+1\right)^{th}$ term.

Therefore, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is $\left(\frac{10}{2}+1\right)^{th} = 6^{th}$ term

$$\begin{aligned} T_6 = T_{5+1} &= {}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5 = \frac{10!}{5!5!}\cdot\frac{x^5}{3^5}\cdot 9^5\cdot y^5 \\ &= \frac{10.9\cdot 8\cdot 7\cdot 6.5!}{5\cdot 4\cdot 3\cdot 2.5!}\cdot\frac{1}{3^5}\cdot 3^{10}\cdot x^5y^5 \qquad \left[9^5 = \left(3^2\right)^5 = 3^{10}\right] \\ &= 252\times 3^5\cdot x^5\cdot y^5 = 61236x^5y^5 \end{aligned}$$

Thus, the middle term in the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ is 61236 x^5y^5.

Question 9:

In the expansion of $(1 + a)^{m + n}$, prove that coefficients of a^m and a^n are equal.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$.

Assuming that a^m occurs in the $(r + 1)^{th}$ term of the expansion $(1 + a)^{m + n}$, we obtain

$T_{r+1} = {}^{m+n}C_r (1)^{m+n-r}(a)^r = {}^{m+n}C_r a^r$

Comparing the indices of a in a^m and in $T_{r + 1}$, we obtain

$r = m$

Therefore, the coefficient of a^m is

$$^{m+n}C_m = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m+n)!}{m!n!} \qquad ...(1)$$

Assuming that a^n occurs in the $(k + 1)^{th}$ term of the expansion $(1 + a)^{m+n}$, we obtain

$$T_{k+1} = {}^{m+n}C_k (1)^{m+n-k} (a)^k = {}^{m+n}C_k (a)^k$$

Comparing the indices of a in a^n and in T_{k+1}, we obtain

$k = n$

Therefore, the coefficient of a^n is

$$^{m+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!m!} \qquad ...(2)$$

Thus, from (1) and (2), it can be observed that the coefficients of a^m and a^n in the expansion of $(1 + a)^{m+n}$ are equal.

Question 10:

The coefficients of the $(r - 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1:3:5. Find n and r.

Answer

It is known that $(k + 1)^{th}$ term, (T_{k+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{k+1} = {}^nC_k a^{n-k} b^k.$$

Therefore, $(r - 1)^{th}$ term in the expansion of $(x + 1)^n$ is

$$T_{r-1} = {}^nC_{r-2} (x)^{n-(r-2)} (1)^{(r-2)} = {}^nC_{r-2} x^{n-r+2}$$

r^{th} term in the expansion of $(x + 1)^n$ is $T_r = {}^nC_{r-1} (x)^{n-(r-1)} (1)^{(r-1)} = {}^nC_{r-1} x^{n-r+1}$

$(r + 1)^{th}$ term in the expansion of $(x + 1)^n$ is $T_{r+1} = {}^nC_r (x)^{n-r} (1)^r = {}^nC_r x^{n-r}$

Therefore, the coefficients of the $(r - 1)^{th}$, r^{th}, and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are ${}^nC_{r-2}$, ${}^nC_{r-1}$, and ${}^nC_r$ respectively. Since these coefficients are in the ratio 1:3:5, we obtain

$$\frac{{}^{n}C_{r-2}}{{}^{n}C_{r-1}}=\frac{1}{3} \text{ and } \frac{{}^{n}C_{r-1}}{{}^{n}C_{r}}=\frac{3}{5}$$

$$\frac{{}^{n}C_{r-2}}{{}^{n}C_{r-1}}=\frac{n!}{(r-2)!(n-r+2)!}\times\frac{(r-1)!(n-r+1)!}{n!}=\frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)(n-r+1)!}$$

$$=\frac{r-1}{n-r+2}$$

$$\therefore \frac{r-1}{n-r+2}=\frac{1}{3}$$

$$\Rightarrow 3r-3=n-r+2$$

$$\Rightarrow n-4r+5=0 \qquad \ldots(1)$$

$$\frac{{}^{n}C_{r-1}}{{}^{n}C_{r}}=\frac{n!}{(r-1)!(n-r+1)}\times\frac{r!(n-r)!}{n!}=\frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)(n-r)!}$$

$$=\frac{r}{n-r+1}$$

$$\therefore \frac{r}{n-r+1}=\frac{3}{5}$$

$$\Rightarrow 5r=3n-3r+3$$

$$\Rightarrow 3n-8r+3=0 \qquad \ldots(2)$$

Multiplying (1) by 3 and subtracting it from (2), we obtain

$4r - 12 = 0$

$\Rightarrow r = 3$

Putting the value of r in (1), we obtain

$n - 12 + 5 = 0$

$\Rightarrow n = 7$

Thus, $n = 7$ and $r = 3$

Question 11:

Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Answer

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^{n}C_{r}a^{n-r}b^{r}$.

Assuming that x^n occurs in the $(r + 1)^{th}$ term of the expansion of $(1 + x)^{2n}$, we obtain

$$T_{r+1} = {}^{2n}C_r (1)^{2n-r} (x)^r = {}^{2n}C_r (x)^r$$

Comparing the indices of x in x^n and in T_{r+1}, we obtain

$r = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is

$${}^{2n}C_n = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2} \qquad ...(1)$$

Assuming that x^n occurs in the $(k + 1)^{th}$ term of the expansion $(1 + x)^{2n-1}$, we obtain

$$T_{k+1} = {}^{2n-1}C_k (1)^{2n-1-k} (x)^k = {}^{2n-1}C_k (x)^k$$

Comparing the indices of x in x^n and T_{k+1}, we obtain

$k = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$ is

$${}^{2n-1}C_n = \frac{(2n-1)!}{n!(2n-1-n)!} = \frac{(2n-1)!}{n!(n-1)!}$$

$$= \frac{2n.(2n-1)!}{2n.n!(n-1)!} = \frac{(2n)!}{2.n!n!} = \frac{1}{2}\left[\frac{(2n)!}{(n!)^2}\right] \qquad ...(2)$$

From (1) and (2), it is observed that

$$\frac{1}{2}\left({}^{2n}C_n\right) = {}^{2n-1}C_n$$

$$\Rightarrow {}^{2n}C_n = 2\left({}^{2n-1}C_n\right)$$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Hence, proved.

Question 12:

Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^2 occurs in the $(r + 1)^{th}$ term of the expansion $(1 + x)^m$, we obtain

$$T_{r+1} = {}^mC_r (1)^{m-r} (x)^r = {}^mC_r (x)^r$$

Comparing the indices of x in x^2 and in T_{r+1}, we obtain

$r = 2$

Therefore, the coefficient of x^2 is ${}^mC_2$.

It is given that the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

$$\therefore {}^mC_2 = 6$$

$$\Rightarrow \frac{m!}{2!(m-2)!} = 6$$

$$\Rightarrow \frac{m(m-1)(m-2)!}{2 \times (m-2)!} = 6$$

$$\Rightarrow m(m-1) = 12$$

$$\Rightarrow m^2 - m - 12 = 0$$

$$\Rightarrow m^2 - 4m + 3m - 12 = 0$$

$$\Rightarrow m(m-4) + 3(m-4) = 0$$

$$\Rightarrow (m-4)(m+3) = 0$$

$$\Rightarrow (m-4) = 0 \text{ or } (m+3) = 0$$

$$\Rightarrow m = 4 \text{ or } m = -3$$

Thus, the positive value of m, for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6, is 4.

NCERT Miscellaneous Solutions

Question 1:

Find *a*, *b* and *n* in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$.

The first three terms of the expansion are given as 729, 7290, and 30375 respectively. Therefore, we obtain

$$T_1 = {}^nC_0 a^{n-0} b^0 = a^n = 729 \qquad ...(1)$$

$$T_2 = {}^nC_1 a^{n-1} b^1 = na^{n-1}b = 7290 \qquad ...(2)$$

$$T_3 = {}^nC_2 a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b^2 = 30375 \qquad ...(3)$$

Dividing (2) by (1), we obtain

$$\frac{na^{n-1}b}{a^n} = \frac{7290}{729}$$

$$\Rightarrow \frac{nb}{a} = 10 \qquad ...(4)$$

Dividing (3) by (2), we obtain

$$\frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} = \frac{30375}{7290}$$

$$\Rightarrow \frac{(n-1)b}{2a} = \frac{30375}{7290}$$

$$\Rightarrow \frac{(n-1)b}{a} = \frac{30375 \times 2}{7290} = \frac{25}{3}$$

$$\Rightarrow \frac{nb}{a} - \frac{b}{a} = \frac{25}{3}$$

$$\Rightarrow 10 - \frac{b}{a} = \frac{25}{3} \qquad [\text{Using } (4)]$$

$$\Rightarrow \frac{b}{a} = 10 - \frac{25}{3} = \frac{5}{3} \qquad \ldots(5)$$

From (4) and (5), we obtain

$$n \cdot \frac{5}{3} = 10$$

$$\Rightarrow n = 6$$

Substituting $n = 6$ in equation (1), we obtain

$a^6 = 729$

$$\Rightarrow a = \sqrt[6]{729} = 3$$

From (5), we obtain

$$\frac{b}{3} = \frac{5}{3} \Rightarrow b = 5$$

Thus, $a = 3$, $b = 5$, and $n = 6$.

Question 2:

Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.

Answer

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

Assuming that x^2 occurs in the $(r + 1)^{th}$ term in the expansion of $(3 + ax)^9$, we obtain

$$T_{r+1} = {}^9C_r (3)^{9-r} (ax)^r = {}^9C_r (3)^{9-r} a^r x^r$$

Comparing the indices of x in x^2 and in T_{r+1}, we obtain

$r = 2$

Thus, the coefficient of x^2 is

$$ {}^9C_2(3)^{9-2}a^2 = \frac{9!}{2!7!}(3)^7 a^2 = 36(3)^7 a^2 $$

Assuming that x^3 occurs in the $(k + 1)^{th}$ term in the expansion of $(3 + ax)^9$, we obtain

$$ T_{k+1} = {}^9C_k(3)^{9-k}(ax)^k = {}^9C_k(3)^{9-k}a^k x^k $$

Comparing the indices of x in x^3 and in T_{k+1}, we obtain

$k = 3$

Thus, the coefficient of x^3 is

$$ {}^9C_3(3)^{9-3}a^3 = \frac{9!}{3!6!}(3)^6 a^3 = 84(3)^6 a^3 $$

It is given that the coefficients of x^2 and x^3 are the same.

$$ 84(3)^6 a^3 = 36(3)^7 a^2 $$
$$ \Rightarrow 84a = 36\times 3 $$
$$ \Rightarrow a = \frac{36\times 3}{84} = \frac{104}{84} $$
$$ \Rightarrow a = \frac{9}{7} $$

Thus, the required value of a is $\frac{9}{7}$.

Question 3:

Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 - x)^7$ using binomial theorem.

Answer

Using Binomial Theorem, the expressions, $(1 + 2x)^6$ and $(1 - x)^7$, can be expanded as

$$ \begin{aligned} (1+2x)^6 &= {}^6C_0 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 \\ &\quad + {}^6C_5(2x)^5 + {}^6C_6(2x)^6 \\ &= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + 15(2x)^4 + 6(2x)^5 + (2x)^6 \\ &= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6 \end{aligned} $$

$$(1-x)^7 = {}^7C_0 - {}^7C_1(x) + {}^7C_2(x)^2 - {}^7C_3(x)^3 + {}^7C_4(x)^4$$
$$- {}^7C_5(x)^5 + {}^7C_6(x)^6 - {}^7C_7(x)^7$$
$$= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$$

$$\therefore (1+2x)^6(1-x)^7$$
$$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6)(1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7)$$

The complete multiplication of the two brackets is not required to be carried out. Only those terms, which involve x^5, are required.

The terms containing x^5 are

$$1(-21x^5) + (12x)(35x^4) + (60x^2)(-35x^3) + (160x^3)(21x^2) + (240x^4)(-7x) + (192x^5)(1)$$
$$= 171x^5$$

Thus, the coefficient of x^5 in the given product is 171.

Question 4:

If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer.

[**Hint:** write $a^n = (a - b + b)^n$ and expand]

Answer

In order to prove that $(a - b)$ is a factor of $(a^n - b^n)$, it has to be proved that

$a^n - b^n = k(a - b)$, where k is some natural number

It can be written that, $a = a - b + b$

$$\therefore a^n = (a-b+b)^n = [(a-b)+b]^n$$
$$= {}^nC_0(a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + {}^nC_n b^n$$
$$= (a-b)^n + {}^nC_1(a-b)^{n-1}b + \ldots + {}^nC_{n-1}(a-b)b^{n-1} + b^n$$
$$\Rightarrow a^n - b^n = (a-b)\left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right]$$
$$\Rightarrow a^n - b^n = k(a-b)$$

where, $k = \left[(a-b)^{n-1} + {}^nC_1(a-b)^{n-2}b + \ldots + {}^nC_{n-1}b^{n-1}\right]$ is a natural number

This shows that $(a - b)$ is a factor of $(a^n - b^n)$, where n is a positive integer.

Question 5:

Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6$.

Answer

Firstly, the expression $(a + b)^6 - (a - b)^6$ is simplified by using Binomial Theorem.

This can be done as

$$\begin{aligned}(a+b)^6 &= {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6b^6\\ &= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6\\ (a-b)^6 &= {}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 - {}^6C_5a^1b^5 + {}^6C_6b^6\\ &= a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\end{aligned}$$

$$\therefore (a+b)^6-(a-b)^6 = 2\left[6a^5b+20a^3b^3+6ab^5\right]$$

Putting $a=\sqrt{3}$ and $b=\sqrt{2}$, we obtain

$$\begin{aligned}\left(\sqrt{3}+\sqrt{2}\right)^6-\left(\sqrt{3}-\sqrt{2}\right)^6 &= 2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right)+20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3+6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]\\ &= 2\left[54\sqrt{6}+120\sqrt{6}+24\sqrt{6}\right]\\ &= 2\times198\sqrt{6}\\ &= 396\sqrt{6}\end{aligned}$$

Question 6:

Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4$.

Answer

Firstly, the expression $(x + y)^4 + (x - y)^4$ is simplified by using Binomial Theorem.

This can be done as

$$(x+y)^4 = {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x-y)^4 = {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 - {}^4C_3xy^3 + {}^4C_4y^4$$
$$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$
$$\therefore (x+y)^4 + (x-y)^4 = 2\left(x^4 + 6x^2y^2 + y^4\right)$$

Putting $x = a^2$ and $y = \sqrt{a^2-1}$, we obtain

$$\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4 = 2\left[\left(a^2\right)^4 + 6\left(a^2\right)^2\left(\sqrt{a^2-1}\right)^2 + \left(\sqrt{a^2-1}\right)^4\right]$$
$$= 2\left[a^8 + 6a^4\left(a^2-1\right) + \left(a^2-1\right)^2\right]$$
$$= 2\left[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1\right]$$
$$= 2\left[a^8 + 6a^6 - 5a^4 - 2a^2 + 1\right]$$
$$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$$

Question 7:

Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Answer

0.99 = 1 – 0.01

$$\therefore (0.99)^5 = (1-0.01)^5$$
$$= {}^5C_0(1)^5 - {}^5C_1(1)^4(0.01) + {}^5C_2(1)^3(0.01)^2 \qquad \text{(Approximately)}$$
$$= 1 - 5(0.01) + 10(0.01)^2$$
$$= 1 - 0.05 + 0.001$$
$$= 1.001 - 0.05$$
$$= 0.951$$

Thus, the value of $(0.99)^5$ is approximately 0.951.

Question 8:

Find *n*, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$

Answer

In the expansion, $(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_{n-1}ab^{n-1} + {}^nC_nb^n$,

Fifth term from the beginning $= {}^nC_4a^{n-4}b^4$

Fifth term from the end $= {}^nC_{n-4}a^4b^{n-4}$

Therefore, it is evident that in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$, the fifth term from the beginning is ${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4$ and the fifth term from the end is ${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$.

$${}^nC_4\left(\sqrt[4]{2}\right)^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4 = {}^nC_4\frac{\left(\sqrt[4]{2}\right)^n}{\left(\sqrt[4]{2}\right)^4}\cdot\frac{1}{3} = {}^nC_4\frac{\left(\sqrt[4]{2}\right)^n}{2}\cdot\frac{1}{3} = \frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n \qquad \ldots(1)$$

$${}^nC_{n-4}\left(\sqrt[4]{2}\right)^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^nC_{n-4}.2\cdot\frac{\left(\sqrt[4]{3}\right)^4}{\left(\sqrt[4]{3}\right)^n} = {}^nC_{n-4}.2\cdot\frac{3}{\left(\sqrt[4]{3}\right)^n} = \frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n} \qquad \ldots(2)$$

It is given that the ratio of the fifth term from the beginning to the fifth term from the end is $\sqrt{6}:1$. Therefore, from (1) and (2), we obtain

$$\frac{n!}{6.4!(n-4)!}\left(\sqrt[4]{2}\right)^n : \frac{6n!}{(n-4)!4!}\cdot\frac{1}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6} : \frac{6}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6}:1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6}\times\frac{\left(\sqrt[4]{3}\right)^n}{6} = \sqrt{6}$$

$$\Rightarrow \left(\sqrt[4]{6}\right)^n = 36\sqrt{6}$$

$$\Rightarrow 6^{\frac{n}{4}} = 6^{\frac{5}{2}}$$

$$\Rightarrow \frac{n}{4} = \frac{5}{2}$$

$$\Rightarrow n = 4\times\frac{5}{2} = 10$$

Thus, the value of n is 10.

Question 9:

Expand using Binomial Theorem $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x \neq 0$.

Answer

Using Binomial Theorem, the given expression $\left(1+\frac{x}{2}-\frac{2}{x}\right)^4$ can be expanded as

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$= {}^4C_0\left(1+\frac{x}{2}\right)^4 - {}^4C_1\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^4C_2\left(1+\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^4C_3\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{2}{x}\right)^4$$

$$=\left(1+\frac{x}{2}\right)^4 - 4\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + 6\left(1+x+\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4}$$

$$=\left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \qquad ...(1)$$

Again by using Binomial Theorem, we obtain

$$\left(1+\frac{x}{2}\right)^4 = {}^4C_0(1)^4 + {}^4C_1(1)^3\left(\frac{x}{2}\right) + {}^4C_2(1)^2\left(\frac{x}{2}\right)^2 + {}^4C_3(1)^1\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4$$

$$=1+4\times\frac{x}{2}+6\times\frac{x^2}{4}+4\times\frac{x^3}{8}+\frac{x^4}{16}$$

$$=1+2x+\frac{3x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16} \qquad ...(2)$$

$$\left(1+\frac{x}{2}\right)^3 = {}^3C_0(1)^3 + {}^3C_1(1)^2\left(\frac{x}{2}\right) + {}^3C_2(1)\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3$$

$$=1+\frac{3x}{2}+\frac{3x^2}{4}+\frac{x^3}{8} \qquad ...(3)$$

From (1), (2), and (3), we obtain

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$=1+2x+\frac{3x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-\frac{8}{x}\left(1+\frac{3x}{2}+\frac{3x^2}{4}+\frac{x^3}{8}\right)+\frac{8}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}+\frac{16}{x^4}$$

$$=1+2x+\frac{3}{2}x^2+\frac{x^3}{2}+\frac{x^4}{16}-\frac{8}{x}-12-6x-x^2+\frac{8}{x^2}+\frac{24}{x}+6-\frac{32}{x^3}+\frac{16}{x^4}$$

$$=\frac{16}{x}+\frac{8}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}-5$$

Question 10:

Find the expansion of $\left(3x^2-2ax+3a^2\right)^3$ using binomial theorem.

Answer

Using Binomial Theorem, the given expression $\left(3x^2-2ax+3a^2\right)^3$ can be expanded as

$$\left[\left(3x^2-2ax\right)+3a^2\right]^3$$
$$={}^3C_0\left(3x^2-2ax\right)^3+{}^3C_1\left(3x^2-2ax\right)^2\left(3a^2\right)+{}^3C_2\left(3x^2-2ax\right)\left(3a^2\right)^2+{}^3C_3\left(3a^2\right)^3$$
$$=\left(3x^2-2ax\right)^3+3\left(9x^4-12ax^3+4a^2x^2\right)\left(3a^2\right)+3\left(3x^2-2ax\right)\left(9a^4\right)+27a^6$$
$$=\left(3x^2-2ax\right)^3+81a^2x^4-108a^3x^3+36a^4x^2+81a^4x^2-54a^5x+27a^6$$
$$=\left(3x^2-2ax\right)^3+81a^2x^4-108a^3x^3+117a^4x^2-54a^5x+27a^6 \qquad \ldots(1)$$

Again by using Binomial Theorem, we obtain

$$\left(3x^2-2ax\right)^3$$
$$={}^3C_0\left(3x^2\right)^3-{}^3C_1\left(3x^2\right)^2\left(2ax\right)+{}^3C_2\left(3x^2\right)\left(2ax\right)^2-{}^3C_3\left(2ax\right)^3$$
$$=27x^6-3\left(9x^4\right)\left(2ax\right)+3\left(3x^2\right)\left(4a^2x^2\right)-8a^3x^3$$
$$=27x^6-54ax^5+36a^2x^4-8a^3x^3 \qquad \ldots(2)$$

From (1) and (2), we obtain

$$\left(3x^2-2ax+3a^2\right)^3$$
$$=27x^6-54ax^5+36a^2x^4-8a^3x^3+81a^2x^4-108a^3x^3+117a^4x^2-54a^5x+27a^6$$
$$=27x^6-54ax^5+117a^2x^4-116a^3x^3+117a^4x^2-54a^5x+27a^6$$