CHAPTER 3 : TRIGONOMETRIC FUNCTIONS-I

We have read about trigonometric ratios in our earlier classes.

Recall that we defined the ratios of the sides of a right triangle as follows :

$\sin\theta = \frac{c}{b}$, $\cos\theta = \frac{a}{b}$, $\tan\theta = \frac{c}{a}$

and $\text{cosec}\,\theta = \frac{b}{c}$, $\sec\theta = \frac{b}{a}$, $\cot\theta = \frac{a}{c}$

We also developed relationships between these trigonometric ratios as $\sin^2\theta + \cos^2\theta = 1$,

$\sec^2\theta = 1 + \tan^2\theta$, $\text{cosec}^2\theta = 1 + \cot^2\theta$

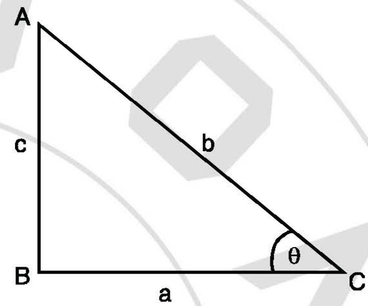


Fig.3.1

We shall try to describe this knowledge gained so far in terms of functions, and try to develop this lesson using functional approach.

In this lesson, we shall develop the science of trigonometry using functional approach. We shall develop the concept of trigonometric functions using a unit circle. We shall discuss the radian measure of an angle and also define trigonometric functions of the type

y = sin x, y = cos x, y = tan x, y = cot x, y = sec x, y = cosec x, y = a sin x, y = b cos x, etc., where *x, y* are real numbers. We shall draw the graphs of functions of the type

y = sin x, y = cos x, y = tan x, y = cotx, y = secx, and y = cosecx y = a sin x, y = a cos x.

OBJECTIVES

After studying this lesson, you will be able to :

- define positive and negative angles;
- define degree and radian as a measure of an angle;
- convert measure of an angle from degrees to radians and vice-versa;
- state the formula $\ell = r\,\theta$ where r and θ have their usual meanings;
- solve problems using the relation $\ell = r\,\theta$;
- define trigonometric functions of a real number;
- draw the graphs of trigonometric functions; and
- interpret the graphs of trigonometric functions.

EXPECTED BACKGROUND KNOWLEDGE

- Definition of an angle.
- Concepts of a straight angle, right angle and complete angle.
- Circle and its allied concepts.
- Special products : $(a \pm b)^2 = a^2 + b^2 \pm 2ab$, $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$
- Knowledge of Pythagoras Theorem and Py thagorean numbers.

3.1 CIRCULAR MEASURE OF ANGLE

An angle is a union of two rays with the common end point. An angle is formed by the rotation of a ray as well. Negative and positive angles are formed according as the rotation is clock-wise or anticlock-wise.

3.1.1 A Unit Circle

It can be seen easily that when a line segment makes one complete rotation, its end point describes a circle. In case the length of the rotating line be one unit then the circle described will be a circle of unit radius. Such a circle is termed as ***unit circle.***

3.1.2 A Radian

A radian is another unit of measurement of an angle other than degree.

A radian is the measure of an angle subtended at the centre of a circle by an arc equal in length to the radius (r) of the circle. In a unit circle one radian will be the angle subtended at the centre of the circle by an arc of unit length.

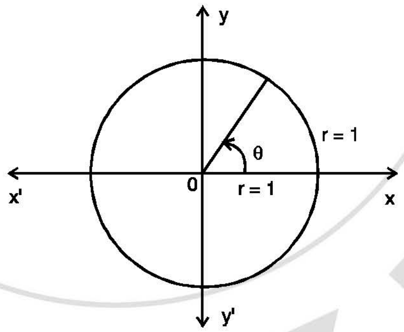


Fig. 3.2

Note : A radian is a constant angle; implying that the measure of the angle subtended by an are of a circle, with length equal to the radius is always the same irrespective of the radius of the circle.

3.1.3 Relation between Degree and Radian

An arc of unit length subtends an angle of 1 radian. The circumference 2π $(\because r = 1)$ subtend an angle of 2π radians.

Hence 2π radians $= 360°$, $\Rightarrow$ π radians $= 180°$, $\Rightarrow$ $\frac{\pi}{2}$ radians $= 90°$

$\Rightarrow \quad \frac{\pi}{4}$ radians $= 45°$ $\quad \Rightarrow \quad$ 1 radian $= \left(\frac{360}{2\pi}\right)^{\circ} = \left(\frac{180}{\pi}\right)^{\circ}$

or $1° = \frac{2\pi}{360}$ radians $= \frac{\pi}{180}$ radians

Example 3.1 Convert

(i) 90° into radians (ii) 15° into radians

(iii) $\frac{\pi}{6}$ radians into degrees. (iv) $\frac{\pi}{10}$ radians into degrees.

Solution :

(i) $1° = \frac{2\pi}{360}$ radians

$\Rightarrow \quad 90° = \frac{2\pi}{360} \times 90$ radians or $90° = \frac{\pi}{2}$ radians

(ii) $15° = \frac{2\pi}{360} \times 15$ radians or $15° = \frac{\pi}{12}$ radians

(iii) 1 radian $= \left(\frac{360}{2\pi}\right)^{\circ}$, $\frac{\pi}{6}$ radians $= \left(\frac{360}{2\pi} \times \frac{\pi}{6}\right)^{\circ}$

$\frac{\pi}{6}$ radians $= 30°$

(iv) $\frac{\pi}{10}$ radians $= \left(\frac{360}{2\pi} \times \frac{\pi}{10}\right)^{\circ}$, $\frac{\pi}{10}$ radians $= 18°$

3.1.4 Relation Between Length of an Arc and Radius of the Circle

An angle of 1 radian is subtended by an arc whose length is equal to the radius of the circle. An angle of 2 radians will be substened if arc is double the radius.

An angle of 2½ radians willbe subtended if arc is 2½ times the radius.

All this can be read from the following table :

Length of the arc (l)	Angle subtended at the centre of the circle θ (in radians)
r	1
2r	2
(2½)r	2½
4r	4

Therefore, $\theta = \frac{\ell}{r}$ or $\ell = r\theta$, where r = radius of the circle,

θ = angle substended at the centre in radians, and ℓ = length of the arc.

The angle subtended by an arc of a circle at the centre of the circle is given by the ratio of the length of the arc and the radius of the circle.

Note : In arriving at the above relation, we have used the radian measure of the angle and not the degree measure. Thus the relation $\theta = \frac{\ell}{r}$ is valid only when the angle is measured in radians.

Example 3.2 Find the angle in radians subtended by an arc of length 10 cm at the centre of a circle of radius 35 cm.

Solution : $\ell = 10cm$ and $r = 35$ cm.

$$\theta = \frac{\ell}{r} \text{ radians} \quad \text{or} \quad \theta = \frac{10}{35} \text{ radians, or} \quad \theta = \frac{2}{7} \text{ radians}$$

Example 3.3 A railroad curve is to be laid out on a circle. What should be the radius of a circular track if the railroad is to turn through an angle of 45° in a distance of 500m?

Solution : Angle θ is given in degrees. To apply the formula $\ell = r\theta, \theta$ must be changed to radians.

$$\theta = 45^\circ = 45 \times \frac{\pi}{180} \text{ radians} \quad ....(1) \quad = \frac{\pi}{4} \text{ radians}$$

$$\ell = 500 \text{ m} \quad ....(2)$$

$$\ell = r\ \theta \text{ gives } r = \frac{\ell}{\theta} \quad \therefore \quad r = \frac{500}{\frac{\pi}{4}} \text{ m} \quad \text{[using (1) and (2)]}$$

$$= 500 \times \frac{4}{\pi} \text{ m}, \ = 2000 \times 0.32 \text{ m} \left(\frac{1}{\pi} = 0.32\right), \ = 640 \text{ m}$$

Example 3.4 A train is travelling at the rate of 60 km per hour on a circular track. Through what angle will it turn in 15 seconds if the radius of the track is $\frac{5}{6}$ km.

Solution : The speed of the train is 60 km per hour. In 15 seconds, it will cover

$$\frac{60 \times 15}{60 \times 60} \text{ km} = \frac{1}{4} \text{ km}$$

$\therefore$ We have $\ell = \dfrac{1}{4}$ km and r $= \dfrac{5}{6}$ km

$$\therefore \quad \theta = \frac{\ell}{r} = \frac{\frac{1}{4}}{\frac{5}{6}} \text{ radians} = \frac{3}{10} \text{ radians}$$

3.2 TRIGONOMETRIC FUNCTIONS

While considering, a unit circle you must have noticed that for every real number between *0* and 2π, there exists a ordered pair of numbers x and y. This ordered pair (x, y) represents the coordinates of the point *P*.

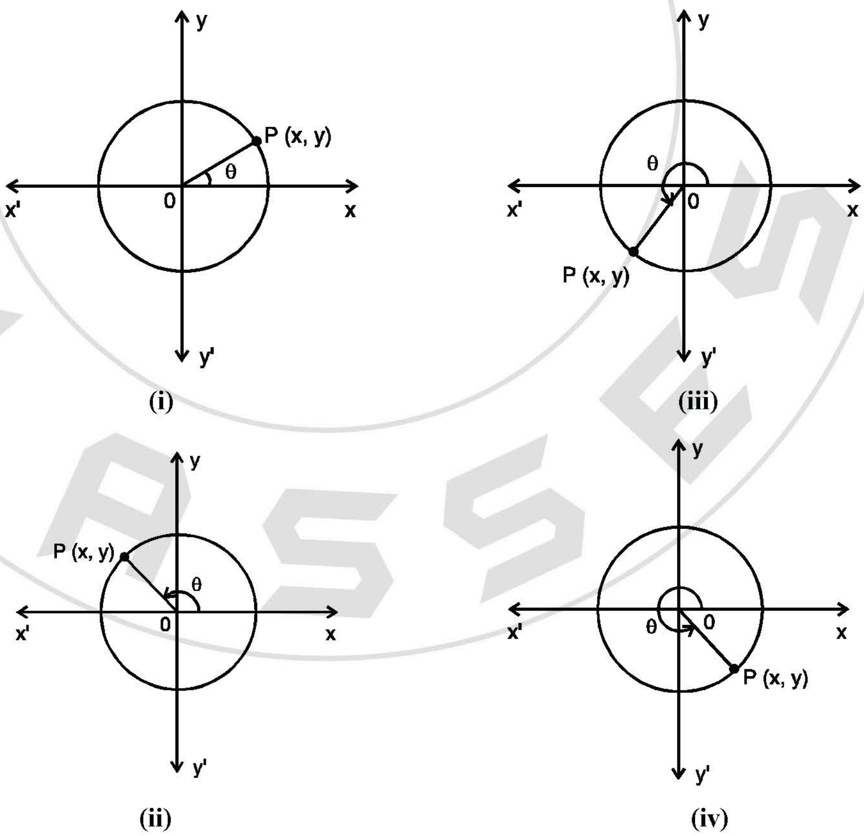


Fig. 3.3

If we consider $\theta = 0$ on the unit circle, we will have a point whose coordinates are (1,0).

If $\theta = \frac{\pi}{2}$, then the corresponding point on the unit circle will have its coordinates (0,1).

In the above figures you can easily observe that no matter what the position of the point, corresponding to every real number θ we have a unique set of coordinates (*x*, *y*). The values of *x* and *y* will be negative or positive depending on the quadrant in which we are considering the point.

Considering a point *P* (on the unit circle) and the corresponding coordinates (*x*, *y*), we define trigonometric functions as :

$$\sin\theta = y,\ \cos\theta = x$$

$$\tan\theta = \frac{y}{x}\ (\text{for } x \neq 0),\ \cot\theta = \frac{x}{y}\ (\text{for } y \neq 0)$$

$$\sec\theta = \frac{1}{x}\ (\text{for } x \neq 0),\ \operatorname{cosec}\theta = \frac{1}{y}\ (\text{for } y \neq 0)$$

Now let the point *P* moves from its original position in anti-clockwise direction. For various positions of this point in the four quadrants, various real numbers θ will be generated. We summarise, the above discussion as follows. For values of θ in the :

I	quadrant, both *x* and *y* are positve.			
II	quadrant, *x* will be negative and *y* will be positive.			
III	quadrant, *x* as well as *y* will be negative.			
IV	quadrant, *x* will be positive and *y* will be negative.			
or	I quadrant	II quadrant	III quadrant	IV quadrant
	All positive	sin positive	tan positive	cos positive
		cosec positive	cot positive	sec positive

Where what is positive can be rememebred by :

	All	sin	tan	cos
Quardrant	I	II	III	IV

If (x, y) are the coordinates of a point P on a unit circle and θ, the real number generated by the position of the point, then $\sin\theta = y$ and $\cos\theta = x$. This means the coordinates of the point P can also be written as ($\cos\theta$, $\sin\theta$)

From Fig. 3.4, you can easily see that the values of x will be between −1 and + 1 as *P* moves on the unit circle. Same will be true for *y* also.

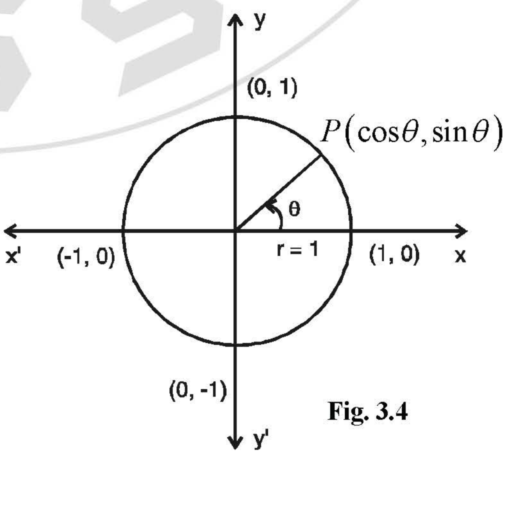


Fig. 3.4

Thus, for all P on the unit circle

$-1 \le x \le 1$ and $-1 \le y \le 1$

Thereby, we conclude that for all real numbers θ

$-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$

In other words, $\sin\theta$ and $\cos\theta$ can not be numerically greater than 1

Example 3.5 What will be sign of the following ?

(i) $\sin\frac{7\pi}{18}$ (ii) $\cos\frac{4\pi}{9}$ (iii) $\tan\frac{5\pi}{9}$

Solution :

(i) Since $\frac{7\pi}{18}$ lies in the first quadrant, the sign of $\sin\frac{7\pi}{18}$ will be posilive.

(ii) Since $\frac{4\pi}{9}$ lies in the first quadrant, the sign of $\cos\frac{4\pi}{9}$ will be positive.

(iii) Since $\frac{5\pi}{9}$ lies in the second quadrant, the sign of $\tan\frac{5\pi}{9}$ will be negative.

Example 3.6 Write the values of (i) $\sin\frac{\pi}{2}$ (ii) $\cos 0$ (iii) $\tan\frac{\pi}{2}$

Solution : (i) From Fig. 3.5, we can see that the coordinates of the point A are $(0,1)$

$\therefore \sin\frac{\pi}{2} = 1$, as $\sin\theta = y$

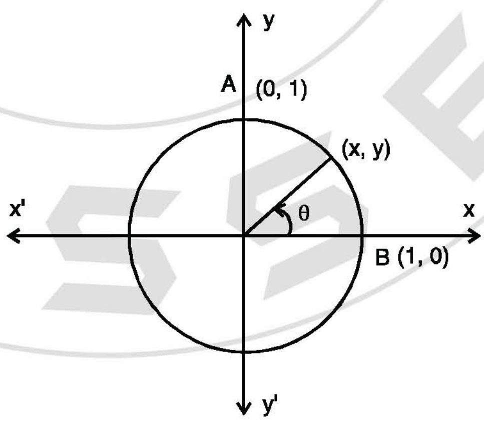


Fig. 3.5

(ii) Coordinates of the point B are $(1, 0)$ $\therefore \cos 0 = 1$, as $\cos\theta = x$

(iii) $\tan\frac{\pi}{2} = \frac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = \frac{1}{0}$ which is not defined , Thus $\tan\frac{\pi}{2}$ is not defined.

Example 3.7 Write the minimum and maximum values of $\cos\theta$.

Solution : We know that $-1 \le \cos\theta \le 1$

$\therefore$ The maximum value of $\cos\theta$ is 1 and the minimum value of $\cos\theta$ is -1.

3.2.1 Relation Between Trigonometric Functions

By definition $x = \cos\theta$, $y = \sin\theta$

As $\tan\theta = \frac{y}{x}, (x \ne 0)$, $= \frac{\sin\theta}{\cos\theta}$, $\theta \ne \frac{n\pi}{2}$

and $\cot\theta = \frac{x}{y}, (y \ne 0)$,

i.e., $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$, $(\theta \ne n\pi)$

Similarly, $\sec\theta = \frac{1}{\cos\theta}$ $\left(\theta \ne \frac{n\pi}{2}\right)$

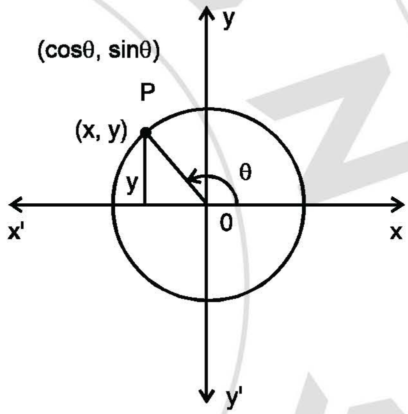


Fig. 3.6

and $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ $(\theta \ne n\pi)$

Using Pythagoras theorem we have, $x^2 + y^2 = 1$, i.e., $(\cos\theta)^2 + (\sin\theta)^2 = 1$

or, $\mathbf{\cos^2\theta + \sin^2\theta = 1}$

Note : $(\cos\theta)^2$ is written as $\cos^2\theta$ and $(\sin\theta)^2$ as $\sin^2\theta$

Again $x^2 + y^2 = 1$ or $1 + \left(\frac{y}{x}\right)^2 = \left(\frac{1}{x}\right)^2$, for $x \ne 0$

or, $1 + (\tan\theta)^2 = (\sec\theta)^2$, i.e. $\mathbf{\sec^2\theta = 1 + \tan^2\theta}$

Similarly, $\mathbf{\operatorname{cosec}^2\theta = 1 + \cot^2\theta}$

Example 3.8 Prove that $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$

Solution : L.H.S. $= \sin^4\theta + \cos^4\theta$

$$= \sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\ \cos^2\theta$$

$$= \left(\sin^2\theta + \cos^2\theta\right)^2 - 2\sin^2\theta\cos^2\theta$$

$$= 1 - 2\sin^2\theta\cos^2\theta\ \left(\because \sin^2\theta + \cos^2\theta = 1\right), = \text{R.H.S.}$$

Example 3.9 Prove that $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

Solution : L.H.S. $= \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\dfrac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}} = \sqrt{\dfrac{(1-\sin\theta)^2}{1-\sin^2\theta}}$

$$= \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}} = \frac{1-\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{R.H.S.}$$

Example 3.10 If $\sin\theta = \dfrac{21}{29}$, prove that $\sec\theta + \tan\theta = -2\dfrac{1}{2}$, given that θ lies in the second quadrant.

Solution : $\sin\theta = \dfrac{21}{29}$ Also, $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow$ $\cos^2\theta = 1 - \sin^2\theta = 1 - \dfrac{441}{841} = \dfrac{400}{841} = \left(\dfrac{20}{29}\right)^2$

$\Rightarrow$ $\cos\theta = \dfrac{-20}{29}$ ($\cos\theta$ is negative as θ lies in the second quardrant)

$\therefore$ $\tan\theta = \dfrac{-21}{20}$ ($\tan\theta$ is negative as θ lies in the second qudrant)

$\therefore$ $\sec\theta + \tan\theta = \dfrac{-29}{20} + \dfrac{-21}{20} = \dfrac{-29-21}{20}, = \dfrac{-5}{2} = -2\dfrac{1}{2} = \text{R.H.S.}$

3.3 TRIGONOMETRIC FUNCTIONS OF SOME SPECIFIC REAL NUMBERS

The values of the trigonometric functions of 0, $\dfrac{\pi}{6}, \dfrac{\pi}{4}, \dfrac{\pi}{3}$ and $\dfrac{\pi}{2}$ are summarised below in the form of a table :

Function ↓ \ Real Numbers → (θ)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

As an aid to memory, we may think of the following pattern for above mentioned values of sin function : , $\sqrt{\frac{0}{4}}, \sqrt{\frac{1}{4}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{3}{4}}, \sqrt{\frac{4}{4}}$

On simplification, we get the values as given in the table. The values for cosines occur in the reverse order.

Example 3.11 Find the value of the following :

(a) $\sin\frac{\pi}{4}\sin\frac{\pi}{3}-\cos\frac{\pi}{4}\cos\frac{\pi}{3}$ (b) $4\tan^2\frac{\pi}{4}-\operatorname{cosec}^2\frac{\pi}{6}-\cos^2\frac{\pi}{3}$

Solution :

(a) $\sin\frac{\pi}{4}\sin\frac{\pi}{3}-\cos\frac{\pi}{4}\cos\frac{\pi}{3} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(b) $4\tan^2\frac{\pi}{4}-\operatorname{cosec}^2\frac{\pi}{6}-\cos^2\frac{\pi}{3}$, $=4(1)^2-(2)^2-\left(\frac{1}{2}\right)^2$, $=4-4-\frac{1}{4}=-\frac{1}{4}$

Example 3.12 If $A=\frac{\pi}{3}$ and $B=\frac{\pi}{6}$, verify that $\cos(A+B)=\cos A\cos B-\sin A\sin B$

Solution : L.H.S. $=\cos(A+B) = \cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\cos\frac{\pi}{2}=0$

R.H.S. $=\cos\frac{\pi}{3}\cos\frac{\pi}{6}-\sin\frac{\pi}{3}\sin\frac{\pi}{6} = \frac{1}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4}=0$

$\therefore$ L.H.S. = 0 = R.H.S.

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

3.4 GRAPHS OF TRIGONOMETRIC FUNCTIONS

Given any function, a pictorial or a graphical representation makes a lasting impression on the minds of learners and viewers. The importance of the graph of functions stems from the fact that this is a convenient way of presenting many properties of the functions. By observing the graph we can examine several characteristic properties of the functions such as (i) periodicity, (ii) intervals in which the function is increasing or decreasing (iii) symmetry about axes, (iv) maximum and minimum points of the graph in the given interval. It also helps to compute the areas enclosed by the curves of the graph.

3.4.1 Variations of sinθ as θ Varies Continuously From 0 to 2π.

Let $X'OX$ and $Y'OY$ be the axes of coordinates. With centre O and radius OP = unity, draw a circle. Let OP starting from OX and moving in anticlockwise direction make an angle θ with the x-axis, i.e. $\angle XOP = \theta$. Draw $PM \perp X'OX$, then $\sin\theta = MP$ as $OP = 1$.

$\therefore$ The variations of $\sin\theta$ are the same as those of MP.

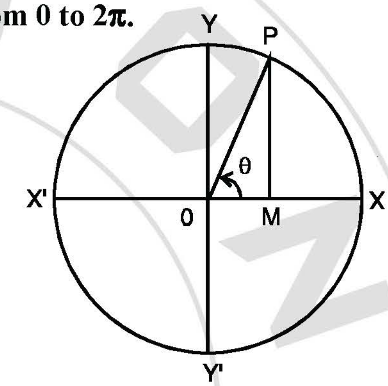


Fig. 3.7

I Quadrant :

As θ increases continuously from 0 to $\frac{\pi}{2}$

PM is positive and increases from 0 to 1.

$\therefore$ $\sin\theta$ is positive.

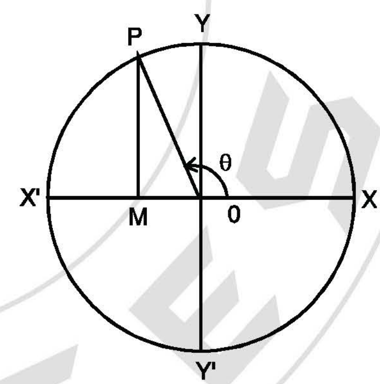


Fig. 3.8

II Quadrant $\left[\frac{\pi}{2}, \pi\right]$

In this interval, θ lies in the second quadrant.

Therefore, point P is in the second quadrant. Here PM = y is positive, but decreases from 1 to 0 as θ varies from $\frac{\pi}{2}$ to π. Thus $\sin\theta$ is positive.

III Quadrant $\left[\pi, \frac{3\pi}{2}\right]$

In this interval, θ lies in the third quandrant. Therefore, point P can move in the third quadrant only. Hence $PM = y$ is negative and decreases from 0 to -1 as θ

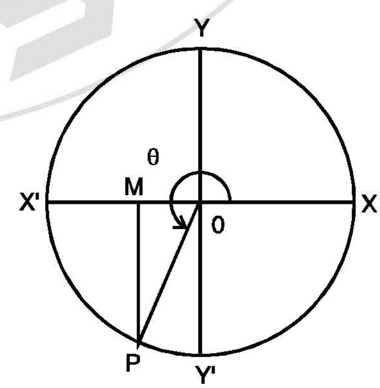


Fig. 3.9

varies from π to $\frac{3\pi}{2}$. In this interval $\sin\theta$ decreases from 0 to -1. In this interval $\sin\theta$ is negative.

IV Quadrant $\left[\frac{3\pi}{2}, 2\pi\right]$

In this interval, θ lies in the fourth quadrant. Therefore, point P can move in the fourth quadrant only. Here again PM = y is negative but increases from -1 to 0 as θ varies from $\frac{3\pi}{2}$ to 2π. Thus $\sin\theta$ is negative in this interval.

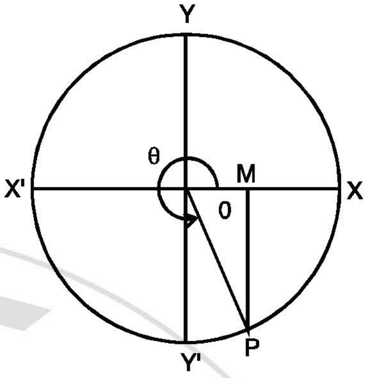


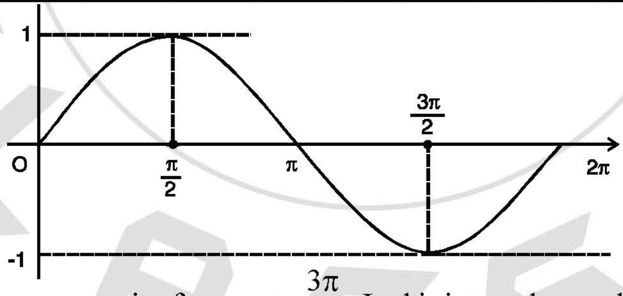
Fig. 3.10

3.4.2 Graph of sin θ as θ varies from 0 to 2π.

Let $X'OX$ and $Y'OY$ be the two coordinate axes of reference. The values of θ are to be measured along x-axis and the values of $\sin\theta$ are to be measured along y-axis.

(Approximate value of $\sqrt{2} = 1.41, \frac{1}{\sqrt{2}} = .707, \frac{\sqrt{3}}{2} = .87$)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	.5	.87	1	.87	.5	0	$-.5$	$-.87$	-1	$-.87$	$-.5$	0



varies from π to $\frac{3\pi}{2}$. In this interval $\sin\theta$ decreases

Fig. 3.11

from 0 to -1. In this interval $\sin\theta$ is negative.

Some Observations

(i) Maximum value of $\sin\theta$ is 1. (ii) Minimum value of $\sin\theta$ is -1.

(iii) It is continuous everywhere. (iv) It is increasing from 0 to $\frac{\pi}{2}$ and from $\frac{3\pi}{2}$ to 2π.

It is decreasing from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. With the help of the graph drawn in Fig. 6.11 we can always draw another graph $y = \sin\theta$ in the interval of $[2\pi, 4\pi]$ (see Fig. 3.12)

What do you observe ?

The graph of $y = \sin\theta$ in the interval $[2\pi, 4\pi]$ is the same as that in 0 to 2π. Therefore, this graph can be drawn by using the property $\sin(2\pi + \theta) = \sin\theta$. Thus, $\sin\theta$ repeats itself when θ is increased by 2π. This is known as the periodicity of $\sin\theta$.

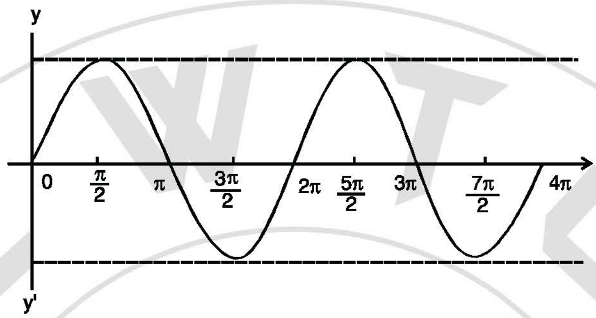


Fig. 3.12

We shall discuss in details the periodicity later in this lesson.

Example 3.13 Draw the graph of $y = \sin 2\theta$ in the interval 0 to π.

Solution :

θ :	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
2θ :	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\sin 2\theta$:	0	.87	1	.87	0	–.87	–1	–.87	0

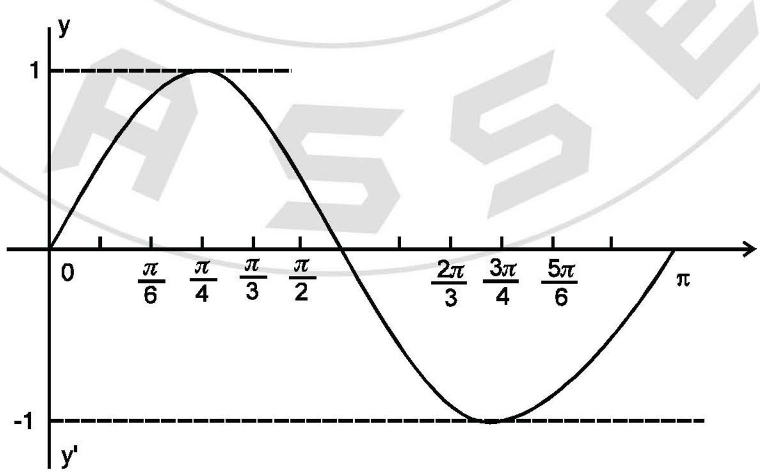


Fig. 3.13

The graph is similar to that of $y = \sin\theta$

Some Observations

1. The other graphs of $\sin\theta$, like $a\sin\theta$, $3\sin 2\theta$ can be drawn applying the same method.
2. Graph of $\sin\theta$, in other intervals namely $[4\pi, 6\pi]$, $[-2\pi, 0]$, $[-4\pi, -2\pi]$, can also be drawn easily. This can be done with the help of properties of allied angles: $\sin(\theta + 2\pi) = \sin\theta$, $\sin(\theta - 2\pi) = \sin\theta$. i.e., θ repeats itself when increased or decreased by 2π.

3.4.3 Graph of $\cos\theta$ as θ Varies From 0 to 2π

As in the case of $\sin\theta$, we shall also discuss the changes in the values of $\cos\theta$ when θ assumes values in the intervals $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$, $\left[\pi, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$.

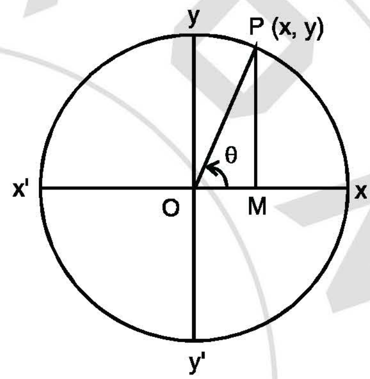


Fig. 3.14

I Quadrant : In the interval $\left[0, \frac{\pi}{2}\right]$, point P lies in the first quadrant, therefore, $OM = x$ is positive but decreases from 1 to 0 as θ increases from 0 to $\frac{\pi}{2}$. Thus in this interval $\cos\theta$ decreases from 1 to 0.

$\therefore$ $\cos\theta$ is positive in this quadrant.

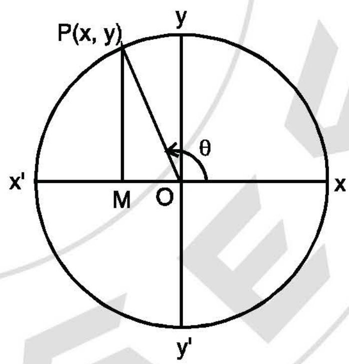


Fig. 3.15

II Quadrant : In the interval $\left[\frac{\pi}{2}, \pi\right]$, point P lies in the second quadrant and therefore point M lies on the negative side of x-axis. So in this case $OM = x$ is negative and decreases from 0 to -1 as θ increases from $\frac{\pi}{2}$ to π. Hence in this inverval $\cos\theta$ decreases from 0 to -1.

$\therefore$ $\cos\theta$ is negative.

III Quadrant : In the interval $\left[\pi, \frac{3\pi}{2}\right]$, point P lies in the third quadrant and therefore, $OM = x$ remains negative as it is on the negative side of x-axis. Therefore $OM = x$ is negative but increases from -1 to 0 as θ increases from π to $\frac{3\pi}{2}$. Hence in this interval $\cos\theta$ increases from -1 to 0.

$\therefore$ $\cos\theta$ is negative.

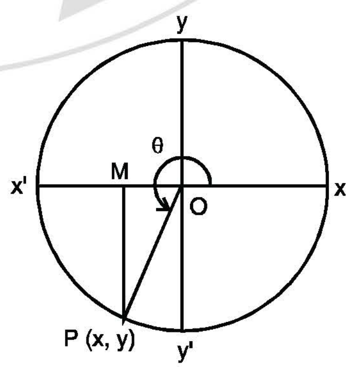


Fig. 3.16

IV Quadrant : In the interval $\left[\frac{3\pi}{2}, 2\pi\right]$, point P lies in the fourth quadrant and M moves on the positive side of x-axis. Therefore $OM = x$ is positive. Also it increases from 0 to 1 as θ increases from $\frac{3\pi}{2}$ to 2π.

Thus in this interval $\cos\theta$ increases from 0 to 1.

$\therefore$ $\cos\theta$ is positive.

Let us tabulate the values of cosines of some suitable values of θ.

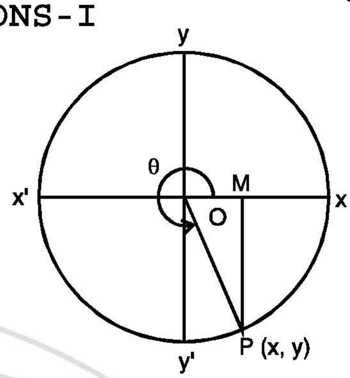


Fig. 3.17

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos\theta$	1	.87	.5	0	0.5	−.87	−1	−.87	−.5	0	0.5	.87	1

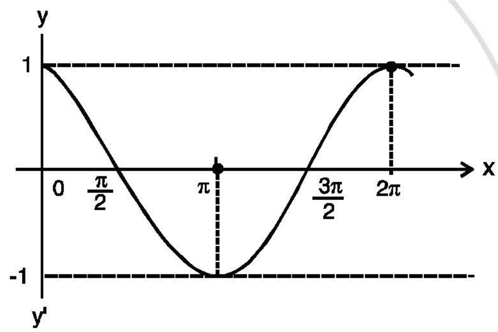


Fig. 3.18

Let $X'OX$ and $Y'OY$ be the axes. Values of θ are measured along x-axis and those of $\cos\theta$ along y-axis.

Some observations

(i) Maximum value of $\cos\theta = 1$. (ii) Minimum value of $\cos\theta = -1$.

(iii) It is continuous everywhere.

(iv) $\cos(\theta + 2\pi) = \cos\theta$. Also $\cos(\theta - 2\pi) = \cos\theta$. Cos θ repeats itself when θ is increased or decreased by 2π. It is called periodicity of $\cos\theta$. We shall discuss in details about this in the later part of this lesson.

(v) Graph of $\cos\theta$ in the intervals $[2\pi, 4\pi]$ $[4\pi, 6\pi]$ $[-2\pi, 0]$, will be the same as in $[0, 2\pi]$.

Example 3.14 Draw the graph of $\cos\theta$ as θ varies from $-\pi$ to π. From the graph read the values of θ when $\cos\theta = \pm 0.5$.

Solution :

θ :	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\cos\theta$:	–1.0	–0.87	–0.5	0	.50	–.87	1.0	0.87	0.5	0	–0.5	–0.87	–1

$$\cos\theta = 0.5$$

when $\theta = \frac{\pi}{3}, \frac{-\pi}{3}$

$$\cos\theta = -0.5$$

when $\theta = \frac{2\pi}{3}, \frac{-2\pi}{3}$

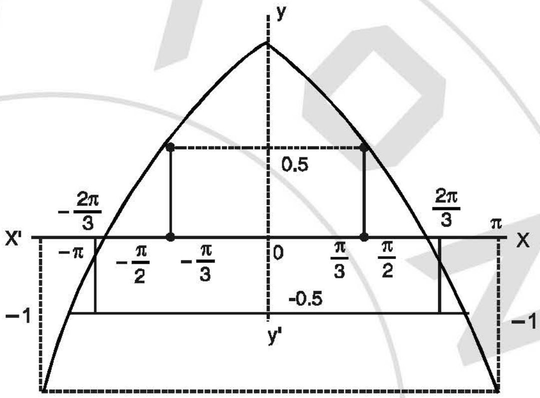


Fig. 3.19

Example 3.15 Draw the graph of $\cos 2\theta$ in the interval 0 to π.

Solution :

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
2θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$\cos 2\theta$	1	0.5	0	–0.5	–1	–0.5	0	0.5	1

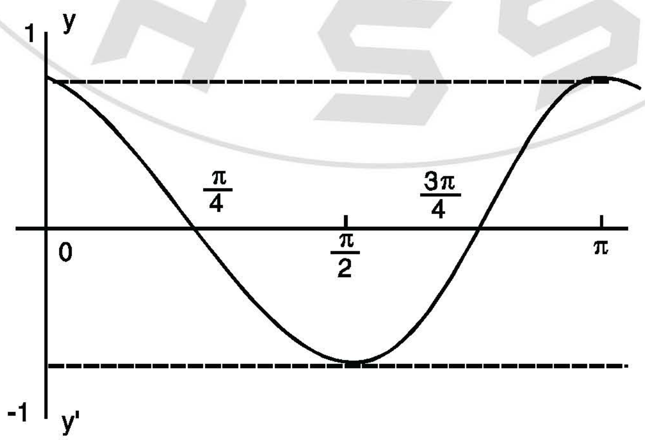


Fig. 3.20

Graph of tan θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}-0°$	$\frac{\pi}{2}+0°$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0°$	$\frac{3\pi}{2}+0°$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
tan θ	0	.58	1.73	$+\infty$	-1.73	-.58	0	.58	1.73	$+\infty$	$-\infty$	-1.73	-.58	0	0

3.4.4 Graph of tan θ as θ Varies from 0 to 2π

In I Quadrant : tan θ can be written as $\frac{\sin\theta}{\cos\theta}$

Behaviour of tan θ depends upon the behaviour of $\sin\theta$ and $\frac{1}{\cos\theta}$

In I quadrant, $\sin\theta$ increases from 0 to 1, $\cos\theta$ decreases from 1 to 0

But $\frac{1}{\cos\theta}$ increases from 1 indefintely (and write it as increasses from 1 to ∞) $\tan\theta > 0$

$\therefore$ tan θ increases from 0 to ∞. (See the table and graph of tan θ).

In II Quadrant : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta$ decreases from 1 to 0.

$\cos\theta$ decreases from 0 to -1.

$\tan\theta$ is negative and increases from $-\infty$ to 0

In III Quadrant : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta$ decreases from 0 to -1

$\cos\theta$ increases from -1 to 0

$\therefore$ $\tan\theta$ is positive and increases from 0 to ∞

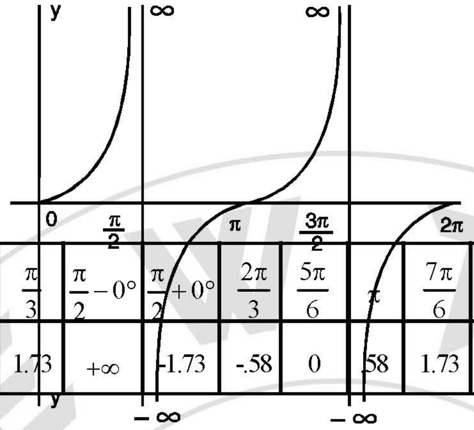
In IV Quadrant : $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta$ increases from -1 to 0

$\cos\theta$ increases from 0 to 1

$\tan\theta$ is negative and increases form $-\infty$ to 0

Graph of tan θ



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}-0^\circ$	$\frac{\pi}{2}+0^\circ$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0^\circ$	$\frac{3\pi}{2}+0^\circ$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\tan\theta$	0	.58	1.73	$+\infty$	-1.73	-.58	0	.58	1.73	$+\infty$	$-\infty$	-1.73	-.58	0	0

Fig. 3.21

Observations

(i) $\tan(180^\circ+\theta)=\tan\theta$. Therefore, the complete graph of $\tan\theta$ consists of infinitely many repetitions of the same to the left as well as to the right.

(ii) Since $\tan(-\theta)=-\tan\theta$, therefore, if $(\theta, \tan\theta)$ is any point on the graph then $(-\theta, -\tan\theta)$ will also be a point on the graph.

(iii) By above results, it can be said that the graph of $y=\tan\theta$ is symmetrical in opposite quadrants.

(iv) $\tan\theta$ may have any numerical value, positve or negative.

(v) The graph of $\tan\theta$ is discontinuous (has a break) at the points $\theta=\frac{\pi}{2}, \frac{3\pi}{2}$.

(vi) As θ passes through these values, $\tan\theta$ suddenly changes from $+\infty$ to $-\infty$.

3.4.5 Graph of cot θ as θ Varies From 0 to 2π

The behaviour of $\cot\theta$ depends upon the behaviour of $\cos\theta$ and $\frac{1}{\sin\theta}$ as $\cot\theta=\cos\theta\frac{1}{\sin\theta}$

We discuss it in each quadrant.

I Quadrant : $\cot\theta=\cos\theta\times\frac{1}{\sin\theta}$

$\cos\theta$ decreases from 1 to 0

$\sin\theta$ increases from 0 to 1

$\therefore$ $\cot\theta$ also decreases from $+\infty$ to 0 but $\cot\theta>0$.

II Quadrant : $\cot\theta = \cos\theta \times \dfrac{1}{\sin\theta}$

$\cos\theta$ decreases from 0 to -1

$\sin\theta$ decreases from 1 to 0

$\Rightarrow$ $\cot\theta < 0$ or $\cot\theta$ decreases from 0 to $-\infty$

III Quadrant : $\cot\theta = \cos\theta \times \dfrac{1}{\sin\theta}$

$\cos\theta$ increases from -1 to 0

$\sin\theta$ decreases from 0 to -1

$\therefore$ $\cot\theta$ decreases from $+\infty$ to 0.

IV Quadrant : $\cot\theta = \cos\theta \times \dfrac{1}{\sin\theta}$

$\cos\theta$ increases from 0 to 1

$\sin\theta$ increases from -1 to 0

$\therefore$ $\cot\theta < 0$

$\cot\theta$ decreases from 0 to $-\infty$

Graph of $\cot\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi-0$	$\pi+0$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cot\theta$	∞	1.73	.58	0	-.58	-1.73	$-\infty$	$+\infty$	1.73	.58	0	-.58	-1.73	$-\infty$

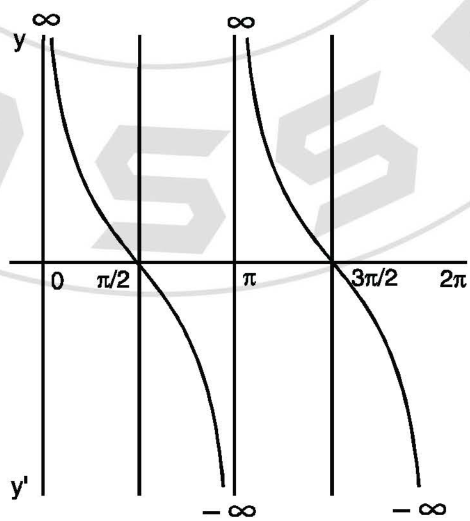


Fig. 3.22

Observations

(i) Since $\cot(\pi+\theta)=\cot\theta$, the complete graph of $\cot\theta$ consists of the portion from $\theta=0$ to $\theta=\pi$ or $\theta=\dfrac{\pi}{2}$ to $\theta=\dfrac{3\pi}{2}$.

(ii) $\cot\theta$ can have any numerical value - positive or negative.

(iii) The graph of $\cot\theta$ is discontinuous, i.e. it breaks at 0, π, 2π, .

(iv) As θ takes values $0, \pi, 2\pi$, $\cot\theta$ suddently changes from $-\infty$ to $+\infty$

3.4.6 To Find the Variations And Draw The Graph of sec θ As θ Varies From 0 to 2π.

Let *X'OX* and *Y'OY* be the axes of coordinates. With centre *O*, draw a circle of unit radius.

Let *P* be any point on the circle. Join *OP* and draw $PM \perp X'OX$.

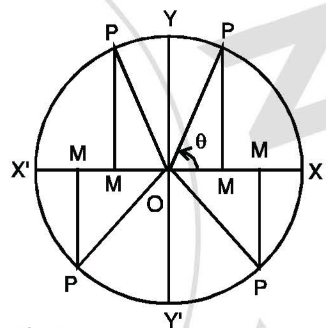


Fig. 3.23

$$\sec\theta=\frac{OP}{OM}=\frac{1}{OM}$$

$\therefore$ Variations will depend upon *OM*.

I Quadrant : $\sec\theta$ is positive as *OM* is positive.

Also $\sec 0=1$ and $\sec\dfrac{\pi}{2}=\infty$ when we approach $\dfrac{\pi}{2}$ from the right.

$\therefore$ As θ varies from 0 to $\dfrac{\pi}{2}$, $\sec\theta$ increases from 1 to ∞.

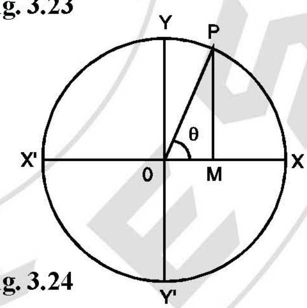


Fig. 3.24

II Quadrant : $\sec\theta$ is negative as *OM* is negative.

$\sec\dfrac{\pi}{2}=-\infty$ when we approach $\dfrac{\pi}{2}$ from the left. Also $\sec\pi=-1$.

$\therefore$ As θ varies from $\dfrac{\pi}{2}$ to π, $\sec\theta$ changes from $-\infty$ to -1.

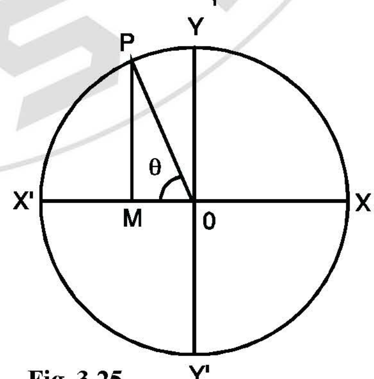


Fig. 3.25

It is observed that as θ passes through $\frac{\pi}{2}$, $\sec\theta$ changes from $+\infty$ to $-\infty$.

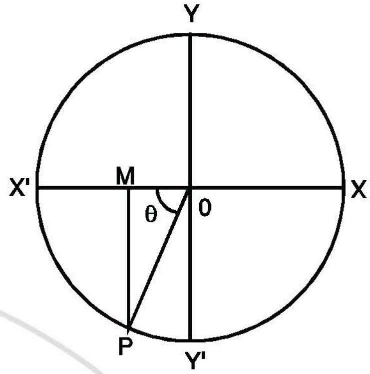


Fig. 3.26

III Quadrant : $\sec\theta$ is negative as OM is negative. $\sec\pi = -1$ and $\sec\frac{3\pi}{2} = -\infty$ when the angle approaches $\frac{3\pi}{2}$ in the counter clockwise direction. As θ varies from π to $\frac{3\pi}{2}$, $\sec\theta$ decreases from -1 to $-\infty$.

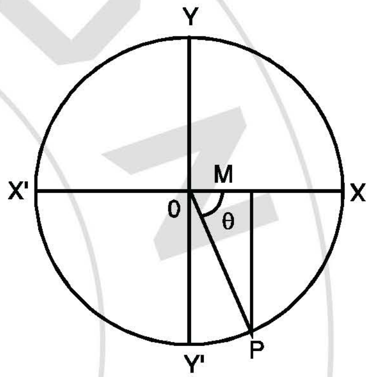


Fig. 3.27

IV Quadrant : $\sec\theta$ is positive as OM is positive. when θ is slightly greater than $\frac{3\pi}{2}$, $\sec\theta$ is positive and very large. Also $\sec 2\pi = 1$. Hence $\sec\theta$ decreases from ∞ to 1 as θ varies from $\frac{3\pi}{2}$ to 2π.

It may be observed that as θ passes through $\frac{3\pi}{2}$; $\sec\theta$ changes from $-\infty$ to $+\infty$.

Graph of sec θ as θ varies from 0 to 2π

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}-0$	$\frac{\pi}{2}+0$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}-0$	$\frac{3\pi}{2}+0$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cot\theta$	1	1.15	2	$+\infty$	$-\infty$	-2	-1.15	-1	-1.15	-2	$-\infty$	$+\infty$	2	1.15	

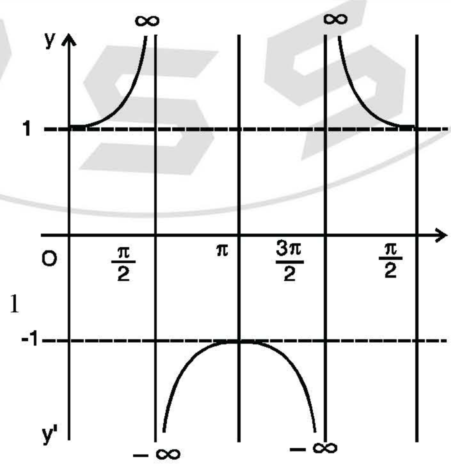


Fig. 3.28

Observations

(a) $\sec\theta$ cannot be numerically less than 1.

(b) Graph of $\sec\theta$ is discontinuous, discontinuties (breaks) occuring at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

(c) As θ passes through $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, see θ changes abruptly from $+\infty$ to $-\infty$ and then from $-\infty$ to $+\infty$ respectively.

3.4.7 Graph of cosec θ as θ Varies From 0 to 2π

Let $X'OX$ and $Y'OY$ be the axes of coordinates. With centre O draw a circle of unit radius. Let P be any point on the circle. Join OP and draw PM perpendicular to $X'OX$.

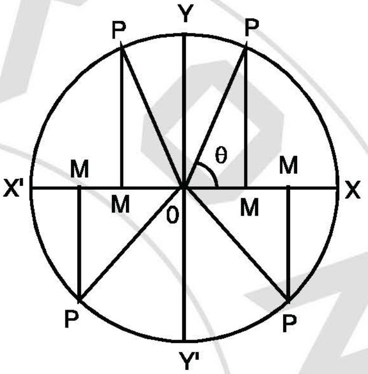


Fig. 3.29

$$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{1}{MP}$$

$\therefore$ The variation of $\operatorname{cosec}\theta$ will depend upon MP.

I Quadrant : cosec θ is positive as MP is positive.

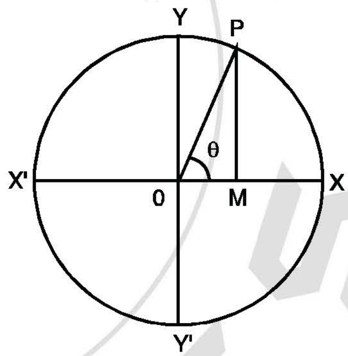


Fig. 3.30

cosec $\frac{\pi}{2} = 1$ when θ is very small, MP is also small and therefore, the value of cosec θ is very large.

$\therefore$ As θ varies from 0 to $\frac{\pi}{2}$, cosec θ decreases from ∞ to 1.

II Quadrant : PM is positive. Therefore, cosec θ is positive. cosec $\frac{\pi}{2} = 1$ and cosec $\pi = \infty$ when the revolving line approaches π in the counter clockwise direction.

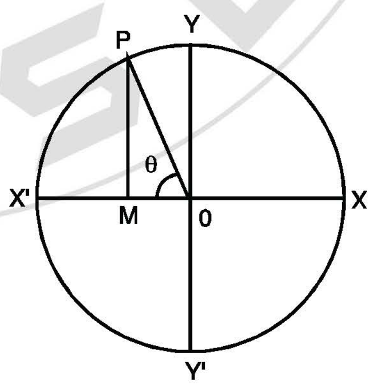


Fig. 3.31

$\therefore$ As θ varies from $\frac{\pi}{2}$ to π, cosec θ increases from 1 to ∞.

III Quadrant : PM is negative

$\therefore$ cosec θ is negative. When θ is slightly greater than π, cosec θ is very large and negative.

Also cosec $\frac{3\pi}{2} = -1$.

$\therefore$ As θ varies from π to $\frac{3\pi}{2}$, cosecθ changes from $-\infty$ to -1.

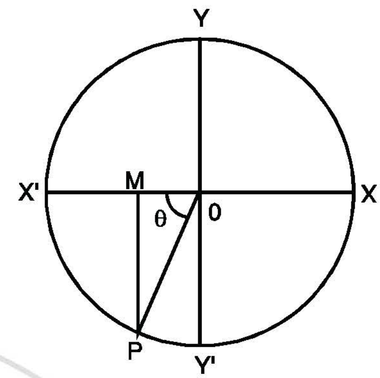


Fig. 3.32

It may be observed that as θ passes through π, cosec θ changes from $+\infty$ to $-\infty$.

IV Quadrant :

PM is negative.

Therefore, cosec $\theta = -\infty$ as θ approaches 2π.

$\therefore$ as θ varies from $\frac{3\pi}{2}$ to 2π, cosecθ varies from -1 to $-\infty$.

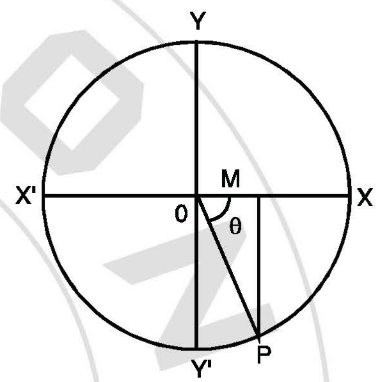


Fig. 3.33

Graph of cosec θ

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\pi-0$	$\pi+0$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
cosec θ	∞	2	1.15	1	1.15	2	$+\infty$	$-\infty$	-2	-1.15	-1	-1.15	-2	$-\infty$

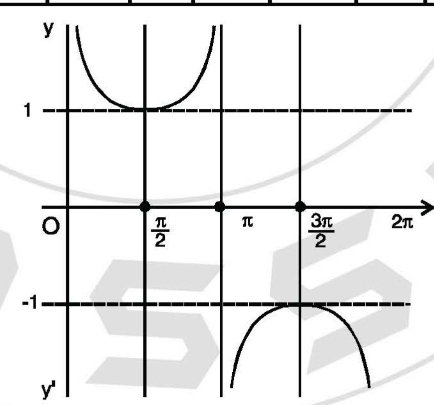


Fig. 3.34

Observations

(a) cosec θ cannot be numerically less than 1.

(b) Graph of cosec θ is discountinous and it has breaks at $\theta = 0, \pi, 2\pi$.

(c) As θ passes through π, cosec θ changes from $+\infty$ to $-\infty$. The values at 0 and 2π are $+\infty$ and $-\infty$ respectively.

Example 3.16 Trace the changes in the values of sec θ as θ lies in $-\pi$ to π.

Soluton :

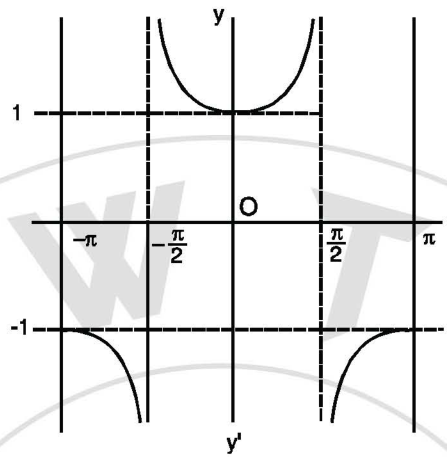


Fig. 3.35

3.5 PERIODICITY OF THE TRIGONOMETRIC FUNCTIONS

From your daily experience you must have observed things repeating themselves after regular intervals of time. For example, days of a week are repeated regularly after 7 days and months of a year are repeated regularly after 12 months. Position of a particle on a moving wheel is another example of the type. The property of repeated occurence of things over regular intervals is known as ***periodicity.***

Definition : A function $f(x)$ is said to be periodic if its value is unchanged when the value of the variable is increased by a constant, that is if $f(x+p)=f(x)$ for all x.

If p is smallest positive constant of this type, then p is called the period of the function $f(x)$.

If $f(x)$ is a periodic function with period p, then $\frac{1}{f(x)}$ is also a periodic function with period p.

3.5.1 Periods of Trigonometric Functions

(i) $\sin x = \sin(x+2n\pi)$; $n = 0, \pm 1, \pm 2, ...$

(ii) $\cos x = \cos(x+2n\pi)$; $n = 0, \pm 1, \pm 2, ...$

Also there is no p, lying in 0 to 2π, for which

$$\sin x = \sin(x+p)$$

$$\cos x = \cos(x+p), \text{ for all } x$$

$\therefore$ 2π is the smallest positive value for which

$\sin(x+2\pi)=\sin x$ and $\cos(x+2\pi)=\cos x$

$\Rightarrow$ sin x and cos x each have the period 2π.

(iii) The period of cosec x is also 2π because cosec $x=\dfrac{1}{\sin x}$.

(iv) The period of sec x is also 2π as sec $x=\dfrac{1}{\cos x}$.

(v) Also $\tan(x+\pi)=\tan x$. Suppose $p\ (0<p<\pi)$ is the period of tan x, then

$\tan(x+p)=\tan x$, for all x. Put x = 0, then tan p = 0, i.e., p = 0 or π.

$\Rightarrow$ the period of tan x is π.

$\therefore$ p can not have values between 0 and π for which $\tan x=\tan(x+p)$

$\therefore$ The period of $\tan x$ is π

(vi) Since $\cot x=\dfrac{1}{\tan x}$, therefore, the period of cot x is also π.

Example 3.17 Find the period of each the following functions :

(a) $y=3\sin 2x$ (b) $y=\cos\dfrac{x}{2}$ (c) $y=\tan\dfrac{x}{4}$

Solution :

(a) Period is $\dfrac{2\pi}{2}$, i.e., π.

(b) $y=\cos\dfrac{1}{2}x$, therefore period $=\dfrac{2\pi}{\frac{1}{2}}=4\pi$

(c) Period of $y=\tan\dfrac{x}{4}=\dfrac{\pi}{\frac{1}{4}}=4\pi$

LET US SUM UP

- An angle is generated by the rotation of a ray.
- The angle can be negative or positive according as rotation of the ray is clockwise or anticlockwise.
- A degree is one of the measures of an angle and one complete rotation generates an angle of 360°.
- An angle can be measured in radians, 360° being equivalent to 2π radians.
- If an arc of length l subtends an angle of θ radians at the centre of the circle with radius r, we have $l = r\theta$.
- If the coordinates of a point P of a unit circle are (x, y) then the six trigonometric functions are defined as $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \frac{y}{x}$, $\cot\theta = \frac{x}{y}$, $\sec\theta = \frac{1}{x}$ and $\operatorname{cosec}\theta = \frac{1}{y}$.

  The coordinates (x, y) of a point P can also be written as $(\cos\theta, \sin\theta)$.

  Here θ is the angle which the line joining centre to the point P makes with the positive direction of x-axis.
- The values of the trigonometric functions $\sin\theta$ and $\cos\theta$ when θ takes values 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ are given by

Real numbers θ → / Functions	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

- Graphs of $\sin\theta$, $\cos\theta$ are continuous every where
  - — Maximum value of both $\sin\theta$ and $\cos\theta$ is 1.
  - — Minimum value of both $\sin\theta$ and $\cos\theta$ is -1.
  - — Period of these functions is 2π.

- $\tan\theta$ and $\cot\theta$ can have any value between $-\infty$ and $+\infty$.

— The function $\tan\theta$ has discontinuities (breaks) at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ in $(0,\ 2\pi)$.

— Its period is π.

— The graph of $\cot\theta$ has discontinuities (breaks) at $0,\ \pi, 2\pi$. Its period is π.

- $\sec\theta$ cannot have any value numerically less than 1.

(i) It has breaks at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. It repeats itself after 2π.

(ii) $\operatorname{cosec}\theta$ cannot have any value between -1 and $+1$.

It has discontinuities (breaks) at $0,\ \pi,\ 2\pi$. It repeats itself after 2π.